NETAJI SUBHAS OPEN UNIVERSITY

STUDY MATERIAL

MATHEMATICS
POST GRADUATE

PG (MT) 05 :
GROUPS A & B

Principles of Mechanics

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Elements of Continuum Mechanics and Special Theory of Relativity

PREFACE

In the curricular structure introduced by this University for students of Post-Graduate degree programme, the opportunity to pursue Post-Graduate course in a subject as introduced by this University is equally available to all learners. Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely in keeping with the objectives of open education which does not believe in artificial differentiation.

Keeping this in view, study materials of the Post-Graduate level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as result of fresh thinking and analysis.

The accepted methodology of distance education has been followed in the preparation of these study materials. Co-operation in every form of experienced scholars is indispensable for a work of this kind. We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing, and devising of a proper lay-out of the materials. Practically speaking, their role amounts to an involvement in invisible teaching. For, whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other.

The more a learner would seriously pursue these study materials the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so that they may be rated as quality self-learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the University.

Needless to add, a great deal of these efforts is still experimental—in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these do admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned.

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বিশ্ববিদ্যালয় মঞ্জুরি কমিশনের দূরশিক্ষা ব্যুরোর বিধি অনুযায়ী মুদ্রিত।
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PG (MT) — 05
Principles of Mechanics
Elements of Continuum Mechanics and Special Theory of Relativity

Group A
Principles of Mechanics

Group
B

Elements of Continuum Mechanics and Special Theory of Relativity

Unit : I ❑ Preliminaries

1.1 A resume of Newton's laws : Newton's three laws of motion date back to 1686, the year they were first proposed in Principia. They may be stated as follows :

I. Everybody continues in its state of rest, or of uniform motion in a straight line, except in so far as it be compelled by external forces to change that state.

II. The rate of change of momentum is proportional to the impressed force and takes place in the direction in which the force acts.

III. To every action there is an equal and opposite reaction.

Newton's first law picks out a class of special frames which are unaccelerated : each one of them moves with a constant velocity with respect to any other one. It is difficult to give a general and broad definition of an inertial frame. A commonly accepted one (but somewhat unpractical) goes as follows : a frame that is fixed relative to the average position of a fixed star or that is moving with a constant velocity (and without any rotation relative to it) is called an inertial frame.

A frame which is not inertial is a non-inertial frame. Thus rotating frames or frames undergoing accelerations are to be treated as non-inertial frames. For short scales (distance and time) the planet Earth serves as an approximate intertial frame. However, strictly speaking, any reference frame attached to Earth has to be rotating (about an axis passing through the geographical poles) and hence ceases to be inertial. We shall later learn that such rotating frames generate fictitious or pseudo forces (eg., the Coriolis force) and that we need to carefully include such forces for using Newton's laws in a non-interial frame.

It is important to realize that Newton's Laws are invariant under transformation

$$\vec{v}(t) \rightarrow \vec{v}'(t) = \vec{v}(t) - \vec{v} \qquad (I.1)$$

where $\vec{v}$ is any constant velocity. (I.1) is called the Galilean law of addition of velocities or simply Galilean transformation. In particular, if $\vec{v}(t)$ is constant then so is $\vec{v}'(t)$ implying that Law I (which states that a body moves at a constant velocity if not acted upon by external forces) is unaffected under the replacement (I. 1). Law I is also called the law of inertia.

Turning to Law II, it states that the force is given by

$$\vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a} \tag{I. 2}$$

where m is the particle mass assumed constant and $\vec{a}$ is the acceleration. Subjecting (I. 2) to the Galilean transformation (I.1)

we find

$$\begin{aligned}\vec{a}' &= \frac{d\vec{v}'(t)}{dt} \\ &= \frac{d}{dt}\left(\vec{v}(t) - \vec{v}\right) \\ &= \frac{d\vec{v}}{dt} \quad (\text{since } \vec{v} \text{ is constant}) \\ &= \vec{a}\end{aligned}$$

Thus Law II, like Law I, is unaffected by (I.1). In this connection it needs to be emphasized that Newton's second law is postulated relative to an inertial frame whose existence is presupposed by the first law.

Law III, which states that actions and reactions are equal and opposite, is also unaffected by change of observers because forces are invariant under a Galilean transformation as just noted above.

Apart from Galilean invariance, Newtonian space and time are homogeneous. It means that every point in the universe is equivalent to every other point and that every moment of time is as good as any other moment. Newtonian space is additionally isotropic, there being no preferential direction. Note that, non-homogeneity in space and time and non-isotropy in space can arise if we are in some accelerating frame. However, as we have already said, Newton's laws are not valid in such a frame.

In classical mechanics the process of obtaining a solution to a given problem involves, in the main, two steps :

(i) choosing an appropriate coordinate system and

(ii) setting up the differential equation as guided by (I.2).

The cartesian system is one of the simplest reference frames to work with. If F_x, F_y, F_z are components of $\vec{F}$ along the x, y, z direction respectively then we have, according to the second law of Newton, $F_x = m\ddot{x}$, $F_y = m\ddot{y}$, $F_z = m\ddot{z}$. However it should be kept in mind that when we switch over to an arbitrary reference frame the transparency of the force-acceleratoin relationships, as provided by the cartesian system, may be lost. As a specific example, we may think of the plane polar co-ordinates (r, θ). The forms of the velocity and acceleration turn out to be $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\,\hat{\theta}$, $\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}$, where $\hat{r}, \hat{\theta}$ are the unit vctors along the directions of $\vec{r}$ and θ. We at once see that $m\ddot{r} \neq F_r$ and $m\ddot{\theta} \neq F_\theta$, F_r, F_θ being respectively the component of the external force in the radial and θ-direction.

However, this does not mean that the use of other co-ordinate systems is discouraged. On the contrary there is a class of problems, for instance the central force, which affords a great deal of simplification if (r, θ) co-ordinates are employed. Some well known examples of the central force are the Newton's inverse square law of gravitation and Coulomb's electrostatic force between two charges. It needs to be pointed out that in a central force problem the underlying force acts in a direction that is towards or away form a fixed point called the 'force centre'. As such the torque $\vec{r} \times \vec{F}$ on the particle about the force centre vanishes resulting in the constancy of the angular momentum $\vec{l} = \vec{r} \times (m\vec{v})$.

Sometimes depending upon the nature of a problem, a spherical polar system with co-ordinates (r, θ, ϕ) or a cylindrical system with co-ordinates (ζ, ϕ, z) proves convenient. These are related to (x, y, z) as

$$x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \; z = r\cos\theta$$

$$x = \zeta \cos\phi, \qquad y = \zeta \sin\phi, \; z = z$$

There also exist other co-ordinate systems such as a parabolic one given by (ξ, η, ϕ). In terms of (x, y, z) these are $x = \xi\eta\cos\phi$, $y = \xi\eta\sin\phi$, $z = \frac{1}{2}\left(\xi^2 - \eta^2\right)$

Problem : Show that the kinetic energy of a particle of mass m in spherical, cylindrical and parabolic system is given by

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) \quad \text{[spherical]}$$

$$T = \frac{1}{2}m\left(\dot{\zeta}^2 + \zeta^2\dot{\phi}^2 + \dot{z}^2\right) \quad \text{[cylindrical]}$$

$$T = \frac{1}{2}\left[\left(\xi^2 + \eta^2\right)\left(\dot{\xi}^2 + \dot{\eta}^2\right) + \xi^2\eta^2\dot{\phi}^2\right] \quad \text{[parabolic]}$$

To set up the differential equation as dictated by (1.2) the procedure involves identifying the guiding forces and making sure that the conditions prescribed in the given problem are the appropriate ones for the Newton's Laws to be applicable. We have already referred to the consideration of an inertial frame. Others are (i) magnitude of the masses of the system and time-distance scales involved should be neither too small (say, those which are comparable to the dimensions of the atomic particles ; here the rules of quantum mechanics apply) nor too big (say, that of the solar system or a galaxy whose dynamics is supposed to be governed by Einstein's general theory of relativity), (ii) magnitude of the velocity must be small compared to the velocity of light c. It is well known that for object moving with high velocities of the order of c the formulas of special theory of relativity come in operation.

1.2 Conservative forces : Conservative forces have a natural occurrence in classical mechanics. Typically, conservative forces are such that the work done by them, as the system moves from one configuration to another, depends only upon the initial and final coordinates of the particles. Conservative forces may be distinguished by any one of the following equivalent features :

(i) The work done by the force is path independent.

(ii) Around any closed path the work done is zero.

(iii) If (F_x, F_y, F_z) are the cartesian components of the force $\vec{F}$, then $F_x\, dx + F_y\, dy + F_z\, dz$ is an exact differential.

(iv) $\vec{F}$ is only a function of position and $\vec{\nabla} \times \vec{F} = 0$.

(v) A potential energy function V exists that has a definite value at every point.

(vi) $T + V =$ constant where T is the kinetic energy.

It may be noted that the criterion (iv) enables one to write $\vec{F}$ as the gradient of some scalar function. Since the gradient points towards the direction of increasing potential and forces cause the system to move towards the lower potential, a negative sign is chosen to express $\vec{F} = -\vec{\nabla} V$: in other words, the force is the negative gradient of some potential function V. In one dimension we have simply

$$F_x = -\frac{dV}{dx}$$

Between the points a and b it gives

$$\int_a^b F_x dx = -\int_a^b dV$$

So work done is $W = -[V(b) - V(a)] = \Delta V$

We therefore have the following principle : work done on the system (positive work) increases potential energy and that work done by the system (negative work) decreases potential energy.

We learnt previously about the constancy of the angular momentum in the central force problem. The orbital plane $\left(\vec{l}\,.\,\vec{r} = 0\right)$ is the one on which the motion lies and which is normal to $\vec{l}$. We assume $\left|\vec{l}\right| \neq 0$. If (r, θ) are the co-ordinates of the particle in the orbital plane with respect to a fixed origin then Newton's second law reads

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = F$$

$$m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = 0 \qquad \text{(I.3)}$$

while the second equation gives $\frac{1}{r}\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$ pointing to the constancy of the angular momentum $l = mr^2\dot{\theta}$ with respect to time, the first equation, on writing $F(r) = -\frac{dV}{dr}$ and eliminating θ, yields

$$m\ddot{r} = -\frac{d}{dr}\left[\frac{l^2}{2mr^2} + V(r)\right] \qquad \text{(I.4)}$$

(I.4) looks like a typical one-dimensional motion (but on the half-line $0 < r < \infty$) influenced by an effective potential

$$\mathrm{U}(l, r) = \frac{l^2}{2mr^2} + V(r) \qquad \text{(I.5)}$$

The first term in the right-hand-side (*rhs*) can be looked upon as coming from a centrifugal force whose magnitude is $\frac{l^2}{mr^3}$.

The total energy that is constant is time is given by

$$\mathrm{E} = \frac{1}{2}m\left|\vec{v}\right|^2 + V(r)$$

$$= \frac{1}{2}m\left|\dot{\vec{r}}\right|^2 + U(l, r) \qquad \text{(I.6)}$$

Some implications of (I.5) and (I.6) will be discussed in Unit III.

Problem : In the gravitational force problem the energy equation is given by

$$E = \frac{1}{2} m \left| \dot{\bar{r}} \right|^2 + \frac{l^2}{2mr^2} - \frac{GMm}{r}$$

for two bodies of masses m and M. Using $\dot{r} = \frac{l}{mr^2}\frac{dr}{d\theta}$, integrate the above equation to arive at an orbit equation in the form of a conic $\frac{\lambda}{r} = 1 + e\cos(\theta - \theta_0)$ where $\lambda = \frac{l^2}{GMm^2}$ is the semi-latus-rectum and $e = \sqrt{1 + \frac{2l^2E}{GM^2m^3}}$ is the eccentricity. Analyse the various cases of the orbit.

1.3 Conservation Laws : This subsection is addessed to the conservation laws for a system of interacting particles.

(a) Conservation of linear momentum :

Separating out the forces acting on, say, the i-th particle $(\bar{r}_i)$ into two parts, one due to the external forces and the remaining due to other particles of the system, we can write the equation of motion as

$$m_i \ddot{\bar{r}}_i = \vec{F}^{i,ext} + \sum_{j \neq i} \vec{F}^{ij} \tag{I.7}$$

where $\vec{F}^{i,ext}$ represents the external force and the second term in the rhs takes care of the force on the i-th particle due to the j-th particle. ; $j \neq i$ implies that self-interactions are ignored.

Summing (I.7) over all the particles in the system we obtain.

$$\sum_i m_i \ddot{\bar{r}}_i = \sum_i \vec{F}^{i,ext} + \sum_i \sum_{j \neq i} \vec{F}^{ij}$$

$$= \vec{\mathfrak{S}} \tag{I.8}$$

where we have made use of Newton's third law : $\vec{F}^{ij} = -\vec{F}^{ij}$ and $\vec{\mathfrak{S}}$ is total external force acting upon the particles.

If the particles are interacting only among themsleves (i.e. not being dictated by the external agency) then $\vec{\mathfrak{S}} = 0$ and (I.8) integrates to $\sum m_i v_i$ a constant vector where $\bar{v}_i = \frac{d\bar{r}_i}{dt}$ is the i-th velocity. We conclude that for a closed system ($\vec{\mathfrak{S}}$ vanishing) the vector sum of the linear momentum of a system of particles is constant in time.

(b) Consevation of angular momentum : Denoting by $\vec{\Omega}$ the resultant angular momentum of a system of particles we can express it as

$$\vec{\Omega} = \sum_i \bar{r}_i \times m_i \bar{v}_i \tag{I.9}$$

where we have summed over the angular momentum of all the particles. Considering now the total time-derivative of $\vec{\Omega}$ we have from (I.9)

$$\frac{d\vec{\Omega}}{dt} = \sum_i \bar{v}_i \times m_i\bar{v}_i + \sum_i \bar{r}_i \times \frac{d}{dt}\left(m_i\bar{v}_i\right)$$

$$= \sum_i \bar{r}_i \times \vec{F}_i \tag{I.10}$$

where we have used (I.2)

For a closed system $\vec{F}_i$ may be replaced by $\sum_{i \neq j} \vec{F}^{ij}$ and so (I.10) becomes

$$\frac{d\vec{\Omega}}{dt} = \sum_i \bar{r}_i \times \sum_{j \neq i} \vec{F}^{ij} \tag{I.11}$$

Clearly the rhs involves typical terms like $\bar{r}_1 \times \bar{F}^{12} + \bar{r}_2 \times \bar{F}^{21}$ which, because of Newtons's third law, acquires the form $\left(\bar{r}_1 - \bar{r}_2\right) \times \bar{F}_{12}$. Now since $\bar{F}_{12}$ acts along the line joining the particles 1 and 2, the vectors product $\left(\bar{r}_1 - \bar{r}_2\right) \times \bar{F}_{12}$ has to vanish. We are thus led to the result that $\frac{d\vec{\Omega}}{dt}$ vanishes for a closed system implying conservation of angular momentum for such systems.

(c) Conservation of energy : the criterion (vi) for the classification of conservative forces follows easily from Newton's second law :

$$\int m\bar{v} \cdot \frac{d\bar{v}}{dt}\, dt = \int \bar{F} . \bar{v}\, dt$$

$$= \int \bar{F} . d\bar{r}$$

$$= -\int \left(\bar{\nabla} V\right) \cdot d\bar{r}$$

$$= -\int dV$$

where V is the potential energy function. Indeed we are led to the conservation of energy :

$$\frac{1}{2} m |\vec{v}|^2 + V = \text{constant} \tag{I.12}$$

Equation (I.12) is readily generalizable to a system of n particles. Consider two particles first. Let us assume that the mutually interacting forces are derivable from a common potential V :

$$\vec{F}_{12} = -\frac{\partial V}{\partial \vec{r}_1}, \quad \vec{F}_{21} = -\frac{\partial V}{\partial \vec{r}_2}$$

Then Newton's second law for these particles reads

$$m_1 \frac{d\vec{v}_1}{dt} = -\frac{\partial V}{\partial \vec{r}_1}, \quad m_2 \frac{d\vec{v}_2}{dt} = -\frac{\partial V}{\partial \vec{r}_2}$$

As a result

$$m_1 \vec{v}_1 \cdot \frac{d\vec{v}_1}{dt} + m_2 \vec{v}_2 \cdot \frac{d\vec{v}_2}{dt} = -\left(\frac{\partial V}{\partial \vec{r}_1} \cdot \frac{d\vec{r}_1}{dt} + \frac{\partial V}{\partial \vec{r}_2} \cdot \frac{d\vec{r}_2}{dt} \right)$$

$$= -\frac{dV}{dt}\left(|\vec{r}_1 - \vec{r}_2|\right),$$

since $F_{12} = -F_{21}$ and F_{12} acts along the line joining particles 1 and 2.

On integrating we have

$$\frac{1}{2} m_1 |\vec{v}_1|^2 + \frac{1}{2} m_2 |\vec{v}_2|^2 + V\left(|\vec{r}_1 - \vec{r}_2|\right) = \text{constant}$$

(I.13) is a generalization of (I.12) for the two-particle system.

To deal with n particles we express Eq (1.8) in the form

$$\sum_i m_i \vec{v}_i \cdot \frac{d\vec{v}_i}{dt} = -\sum_i \left[\frac{\partial}{\partial \vec{r}_i} \left(V^{ext} + V^{int} \right) \cdot \frac{d\vec{r}_i}{dt} \right]$$

where the superscripts (ext) and (int) on V indicate the assumptions that the external forces and forces for the interacting particles are conservative. A simple integration produces the conservation of energy for the n-particle system

$$T + V^{ext} + V^{int} = \text{constant}$$

where T stands for the total kinetic energy for the n particles :

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i |\vec{v}_i|^2$$

I.4 Period of oscillations : Consider the conservation of energy in one-dimension

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + V(x) = E$$

It can be expressed in the form

$$t = \sqrt{\frac{m}{2}}\int \frac{dx}{\sqrt{E - V(x)}} + \text{constant} \qquad (I.14)$$

For a moving particle its kinetic energy is always a positive definite quantity and as such there is the constraint $V(x) < E$. This has the following implication.

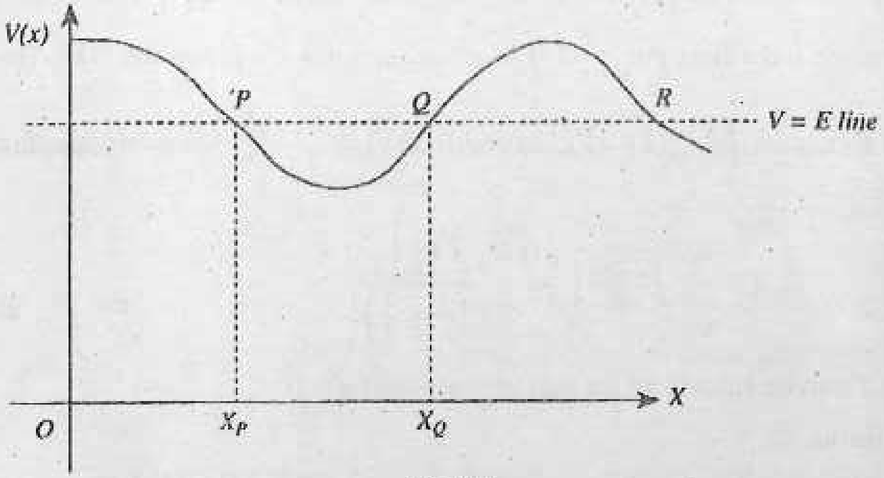


Fig I.1

In Figure I.1, given some profile of $V(x)$, the motion is restricted in the region PQ or in the region to the right of R. The turning points are obtained by solving the equation $V(x) = E$ which prescribe the limits of the motion. It is easy to realize, for the motion confined to the portion PQ (i.e., a finite region), that the particle moves back and forth between the points X_P and X_Q (i.e. oscillatory). The period T of the oscillation is given by the time that the particle takes to travel from X_P to X_Q and back. By symmetry this is twice the time from X_P to X_Q and so from (I.14)

$$T = \sqrt{2m}\int_{X_P}^{X_Q} \frac{dx}{\sqrt{E - V(x)}} \qquad (I.15)$$

where X_P, X_Q are the roots of the equation $V(x) = E$ assumint E to be given.

As a specific application of (1.15) consider $V(x) = A|x|^n$, where A is a positive constant. We get,

$$T = 2\sqrt{2m}\int_0^{(E/A)^{1/2}} \frac{dx}{\sqrt{E - Ax^n}}$$

$$= 2\sqrt{\frac{2m}{E}}\left(\frac{E}{A}\right)^{1/n}\int_0^1 \frac{dz}{\sqrt{1 - z^n}}$$

$$= \frac{2}{n}\sqrt{\frac{2m}{E}}\left(\frac{E}{A}\right)^{1/n}\int_0^1 \frac{dt}{t^{1-\frac{1}{n}}(1-t)^{1/2}}$$

where we have first put $x = \left(\frac{E}{A}\right)^{1/n} z$ and then $t = z^n$. Since the Beta function is given by $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$ and $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, it emerges that

$$T = \frac{2}{m}\sqrt{\frac{2\pi m}{E}}\left(\frac{E}{A}\right)^{1/n} \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$$

Thus T can be calculated for various values of $n > 0$.

Problems :

1. Find the period of oscillation for the potential $V = -V_0 \operatorname{sech}^2 \alpha x$ where $-V_0 < E < 0$. $\left[\text{Answer: } T = \frac{\pi}{\alpha}\sqrt{\frac{2m}{|E|}}\right]$

2. Fid the period of oscillation for the potential

$V = V_0 \tan^2 \alpha x$ $\left[\text{Answer: } T = \frac{\pi}{\alpha}\sqrt{\frac{2m}{E+V_0}}\right]$

Unit : II ❑ Constraints, generalized coordinates and D'Alembert's principle

II.1 Constraints and generalized coordinates : In any physical system the particles usually have their coordinates restricted in one way or the other. For example, in the case of a simple pendulum, the position of the bob obeys equations of the type $x^2 + z^2 = l^2$, $y = 0$ where l is the length of the string and the motion is confined to the vertical xz-plane. As another example, one may consider the sliding of a block along the line of greatest slope on the surface whose equation is $y = mx + c$. The rigid body, for which the distance between any two particles is always fixed, provides one more example. We thus see that the constraints dictate the availability of freedom to a system. The "degree of freedom" is defined as the number of independent coordinates (not including time) required to specify completely the position and configuration of the system. In the three examples cited above, the degree of freedom for the plane pendulum and the sliding of the block on a surface is one while for a rigid body it is six.

Let us consider a system composed of N particles. Newton's law may be written as

$$m_i\ddot{\bar{r}}_i = \bar{F}_i, \quad i = 1, 2, \ldots N \tag{II.1}$$

Since each particle may be specified by three co-ordinates, we have at hand $3N$ coordinates which may be subjected to, say, k ($\leq 3N$) equations of constraints. The number of degrees of freedom is then given by $n = 3N - k$.

A constraint is some kind of restriction on the motion of a particle and the forces responsible for the restriction are called the forces of constraint. The forces of constraint are initially unknown and are required to be solved for. For instance, the tension of the string in the plane pendulum problem is the force of constraint which is to be determined by solving the equation of motion. If the forces other than the forces of constraint are designated as applied forces then $\bar{F}$ appearing in (II.1) may be split up in the manner

$$m_i\ddot{\bar{r}}_i = \bar{F}_i^{(a)} + \bar{F}_i^{(c)}, \; i = 1, 2, \ldots N \tag{II.2}$$

where $\vec{F}_i^{(a)}$ are the applied forces and $\vec{F}_i^{(c)}$ are the constraint forces.

Generally, a constraint is of the form

$$\vec{f}\left(\vec{r}_j, \vec{v}_j, t\right) = 0, \; j = 1, 2, \ldots N \tag{II.3}$$

where $\vec{v}_j \equiv \dot{\vec{r}}_j$ are the velocities. In the absence of velocities, (II.3) is called a finite or geometric constraint :

$$\vec{f}\left(\vec{r}_j, t\right) = 0, \; j = 1, 2, \ldots N \tag{II.4}$$

Otherwise, (II.3) is a differential or kinematical constraint.

A particular class of differential constraints is the linear form

$$\sum_{j=1}^{N} \vec{a}_j \cdot \vec{v}_j + D = 0 \tag{II.5}$$

where $\vec{a}_j$ are not all vanishing and D is a scalar function of $\vec{r}$ and t.

In the stationary case, $\frac{\partial f}{\partial t} = 0$ in (II.4) while in (II.5) $D = 0$ and $\vec{a}_j$ are functions of position only. A system is called scleronomic if only stationary constraints are present ; otherwise it is called rheonomic.

Apart from the type (II. 3) which are called bilateral constraints, there can also be unilateral constraints which appear as inequalities.

$$f\left(\vec{r}_j, \vec{v}_j, t\right) \geq 0 \tag{II.6}$$

As an example of (II.6), one can think of a volume of gas confined in a box of lengths a, b and c. Then the motion of the gas penticles is restricted by $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$.

Constraints may be further distringuished into two classes : the ones that can be expressed as an algetraic equation involving the co-ordinates. Such constraints are holonomic constraints. The corresponding system is a holonomic system. On the other hand, there exists a larger class of constraints which are non-integrable in nature and certainly not expressible as an algebraic equation involving the coordinates. Such constraints are classified as non-holonomic constraints and the system subjected to them is a non-holonomic system.

Some examples given below will help us to clasify the above issues.

Ex. 1. For the simple plane pendulum the equations of constraints are $x^2 + z^2 = l^2$, $y = 0$ where l is the length of the string and the pendulum bob is restricted to swing in the xz-plane.

Here the constraint is holonomic and scleronomic. If however the length of the pendulum changes with time due to seasonal effects then we have a time-dependent (rheonomic) constraint.

Ex. 2. Let a rod of length l connect two particles on a plane which move in such a manner that the vrelocity of the centre of the rod is in the direction of the rod. According to the problem the constraint equations are, apart from $z_1 = z_2 = 0$,

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = l^2$$

$$\frac{\dot{x}_1 + \dot{x}_2}{x_1 - x_2} = \frac{\dot{y}_1 + \dot{y}_2}{y_1 - y_2}$$

where (x_1, y_1), (x_2, y_2) refer to the coordinates of the end points of the rod. Here we run into a non-integrable differential constraint (see the last form) implying that the system under consideration is a non-holonomic one.

Ex. 3. We remarked earlier that the volume of a gas confined in a box of lengths a, b, c is subjected to the unilateral constraints $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. Here the constraints are obviously non-holonomic.

Ex. 4. Constraints may be deceptive such as the following one :

$$(y + yz - 1)\dot{x} + (x + xz - 1)\dot{y} + xy\dot{z} = 0$$

We observe that we can actually integrate the above form and obtain an algebraic equation involving the coordinates :

$$(1 + z)xy = x + y + c$$

where c is an arbitrary constant. The constraint is therefore to be looked upon as a holonomic, scleronomic constraint.

Once the constraint equations have been correctly identified it proves useful to set up a set of n independent variables q_i $(i = 1, 2, \ldots n)$, called the generalized coordinates, to describe the configuration of a physical system. These generalized coordinates, whose total number equals the number of degrees of freedom available for the system, are quite general in character and need not always conform to any of the special types like the cartesian or the polar or say, the parabolic coordinates.

For the plane pendulum problem, see Fig II.1, where the origin O is taken at the point of suspension, the constraints are $x^2 + z^2 = l^2$, $y = 0$. Here any one of x or z or θ (the angle which the string makes with the vertical) may serve as the generalized coordinate.

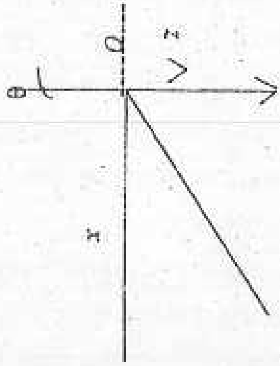


Fig. II. 1

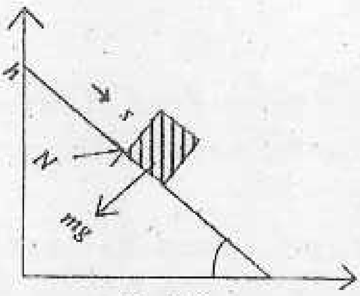


Fig. II. 2

In the case of a block sliding on an inclined plane, see Fig II.2, the distance s from the top down the plane is a good candidate for the generalized coordinate. The relationships between the cartesion coordinates (x, y) and s are given by $x = s \cos \alpha$, $y = h - s \sin \alpha$.

It may be mentioned here that there is no general rule for the adoptation of a particular set of generalized coordinates. Which one to be employed depends a great deal upon an educated guess and also upon the conditions of a problem.

In the following section let us derive an expression for the kinetic energy in terms of the generalized coordinates $q_1, q_2, \ldots q_n$.

II.2 Kinetic energy of a holonomic system :

By definition, the kinetic energy is given by

$$T = \frac{1}{2}\sum_{j=1}^{N} m_j \left|\vec{r}_j\right|^2 \quad j = 1, 2, \ldots N \tag{II.7}$$

where $\vec{r_j}$ is

$$\vec{r_j} = \vec{r_j}\ (q_1, q_2, \dots q_n\ ;\ t)$$

It follows that

$$\dot{\vec{r_j}} = \sum \frac{\partial \vec{r_j}}{\partial q_i}\dot{q}_i + \frac{\partial \vec{r_j}}{\partial t} \qquad \text{(II.8)}$$

Substituting the expression (II.8) into (II. 7) we are led to the following form for T

$$T = \frac{1}{2}\sum_{i,k=1}^{n} a_{ik}\,\dot{q}_i\,\dot{q}_k + \sum_{i=1}^{n} a_i\,\dot{q}_i + a_o \qquad \text{(II.9)}$$

where the coefficients a_{ik}, a_i, a_0 are

$$a_{ik} = \sum_{j=1}^{N} m_j \frac{\partial \vec{r_j}}{\partial q_i}\cdot\frac{\partial \vec{r_j}}{\partial q_k}$$

$$a_i = \sum_{j=1}^{N} m_j \frac{\partial \vec{r_j}}{\partial q_i}\cdot\frac{\partial \vec{r_j}}{\partial t}$$

$$a_0 = \frac{1}{2}\sum_{j=1}^{N} m_j\left(\frac{\partial \vec{r_j}}{\partial t}\right)^2 \qquad \text{(II.10)}$$

and $i, k = 1, 2, \dots n$.

An interesting off-short of (II.9) is that, in the scleronomic case, both the coefficients a_i and a_0 drop out and we are left with a homogeneous function of the second degree of the generalized velocities for the kinetic energy function :

$$T = \frac{1}{2}\sum_{i,k=1}^{n} a_{ik}\,\dot{q}_i\,\dot{q}_k \equiv T_2(say) \qquad \text{(II. 11)}$$

Actually T_2 can be shown to be always degenerate :

$$\det\left(a_{ik}\right)_{i,\ k=1}^{n} \neq 0 \qquad \text{(II. 12)}$$

For if the above determinant were to vanish we would be faced with a set of homogeneous linear equations

$$\sum_{k=1}^{n} a_{ik}\, x_k = 0, \;\; i = 1, 2, \ldots n \tag{II.13}$$

which has a non-zero real solution. Now multiplying the lef-hand-side (*lhs*) of (II.13) by x_i and summing over i we obtain,

$$0 = \sum_{i,k=1}^{N} a_{ik}\, x_i\, x_k = \sum_{j=1}^{N} m_j \left(\sum_{i=1}^{n} x_i \frac{\partial \vec{r}_j}{\partial q_i} \right)^2$$

where (II.10) has been used. Therefore we conclude that

$$\sum_{i=1}^{n} x_i \frac{\partial \vec{r}_j}{\partial q_i} = 0, \;\; j = 1, 2, \ldots \mathrm{N} \tag{II.14}$$

Since $\vec{r}_j \equiv \left(x_j,\, y_j,\, z_j\right)$, (II.14) reflects that the columns of the following Jacobian matrix are linearly dependent :

$$[J] \equiv \begin{bmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} & \cdots & \frac{\partial x_1}{\partial q_n} \\ \frac{\partial y_1}{\partial q_1} & \frac{\partial y_1}{\partial q_2} & \cdots & \frac{\partial y_1}{\partial q_n} \\ \frac{\partial z_1}{\partial q_1} & \frac{\partial z_1}{\partial q_2} & \cdots & \frac{\partial z_1}{\partial q_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial X_N}{\partial q_1} & \frac{\partial X_N}{\partial q_2} & \cdots & \frac{\partial X_N}{\partial q_n} \\ \frac{\partial Y_N}{\partial q_1} & \frac{\partial Y_N}{\partial q_2} & \cdots & \frac{\partial Y_N}{\partial q_n} \\ \frac{\partial Z_N}{\partial q_1} & \frac{\partial Z_N}{\partial q_2} & \cdots & \frac{\partial Z_N}{\partial q_n} \end{bmatrix}$$

In other words the rank ζ of $[J]$ is less than n. That is, of the $3N$ coordinates specified by (x_1, y_1, z_1), (x_2, y_2, z_2), ... (X_N, Y_N, Z_N) there can be only ζ independent quantities in terms of which the remaining ones can be expressed. Thus we arrive at a contradiction since according to the definition of generalized co-ordinates we have n degrees of freedom and $\zeta < n$. Hence (II. 12) holds. Further $T_2 \geq 0$ since T_2 stands for the kinetic energy in the stationary situation. Actually $T_2 > 0$: the equality occurring when $\dot{q}_i = 0$ (i = 1, 2, ... n)

II.3 Virtual work and D'Alembert's principle : Before embarking upon the concept of virtual work, let us consider typical situations when the forces of constraint do work.

First consider the case of simple plane pendulum whose length l is constant. It is obvious that as it swings, the bob traces out an arc of a circle in the xz-plane due to the constraint $x^2 + z^2 = l^2$. The displacement of the bob is normal to the direction of the force of constraint namely the tension T which acts along the length fo the string. Hence work done by the force of constraint is zero.

Next consider the sliding of the block along a frictionless inclined plane. The applied force is the force of gravity while the normal reaction $\vec{N}$ is the force of constraint. Here too, the force of constraint acts perpendicular to the direction towards which the block slides. Accordingly the force of constraint does no work.

Now take the case of a rigid body. Work done by the i-th particle of the rigid body is

$$W_i = \sum_j \vec{F}_{ij} . d\vec{r}_i \quad (i \neq j) \qquad \text{(II. 15)}$$

where $d\vec{r}_i$ stands for the displacement and $\vec{F}_{ij}$ represents the constraint force on the i-th paricle due to the j-th particle. Note that in (II. 15) we have ignored self-forces.

Considering all the particles in the rigid body, we sum over i in (II.15) to get for the total work done

$$W = \sum_i W_i = \sum_i \sum_j \vec{F}_{ij} \cdot d\vec{r}_i \qquad \text{(II.16)}$$

Interchanging i and j in (II.16) and explaiting Newton's third law,

$\vec{F}_{ij} = -\vec{F}_{ji}$, we arrive at

$$W = \frac{1}{2} \sum_i \sum_j \vec{F}_{ij} \cdot \left(d\vec{r}_i - d\vec{r}_j \right) \qquad \text{(II.17)}$$

For a rigid body since the distance between any two particles is fixed we have $\left|\vec{r}_i - \vec{r}_j\right|$ = constant i.e., $\left(\vec{r}_i - \vec{r}_j\right)^2$ = constant. On differentiation it yields $\left(\vec{r}_i - \vec{r}_j\right) \cdot \left(d\vec{r}_i - d\vec{r}_j\right) = 0$. From Newton's third law, $\vec{F}_{ij}$ acts closing $\vec{r}_i - \vec{r}_j$ which is the direction along the line joining the particles i and j. It is thus implied from (II. 17) that $W = 0$ and so the total work done by the forces of constraint in a rigid body is zero.

In all the above examples we found that the total work done by the force (s) of constraint is zero.

However, we run into a difficulty in the time-changing scenarios. Only one example will suffice. Consider the case of the simple pendulum whose length l is changing with time : $l = l(t)$. Here it is obvious that the bob traces out a different route than the usual circular arc. The displacement of the bob is therefore not normal to the direction of T, the force of constraint. So the work done by T is non-zero. We thus see that the forces of constraint can do work if the constraint is time-dependent.

How to treat to time-independent as well as the time-varying constraints in a consistent framework ? Fortunately we have a way out. We invoke the concept of 'virtual' displacement, the word 'virtual' has the underlying meaning that no passage of 'real' time is involved during the displacements taken. In other words, we 'freeze' the system at a certain point of time and think of virtual displacements $\delta\vec{r}_i$ (i = 1, 2, ...) that are consistent with the conditions of the constraint instead of the real ones $d\vec{r}_i$. Some arbitrariness is, of course, involved in the choice of the direction of $\delta\vec{r}_i$. This is exploited by the principle of virtual work stated below :

The total work done by the forces of bilateral constraint in a virtual displacement is zero :

$$\delta W^c = \sum_{j=1}^{N} \vec{F}_j^c \cdot \delta\vec{r}_j = 0 \qquad \text{(II.18)}$$

So from the equations motion (II.2) we are led to

$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \cdot \delta\vec{r}_i = \sum_{i=1}^{N} \vec{F}_i^a \cdot \delta\vec{r}_i + \sum_{i=1}^{N} \vec{F}_i^c \cdot \delta\vec{r}_i$$

which because of (II. 18) rduces to

$$\sum_{i=1}^{N} (\vec{F}_i^a - m_i \ddot{\vec{r}}_i) \cdot \delta \vec{r}_i = 0 \qquad \text{(II. 19)}$$

(II. 19) is the D'Alembert's principle. It asserts that the work done by the applied forces along with the work done by the inertial forces in a virtual displacement is zero. Note that in the principle of virtual work the constraint forces are not associated with frictional forces. Perhaps the biggest success of D'Alembert's principle is its ability to get rid of the constraint foces. However, unlike the Newtonian equations of motion, this principle offers a single equation whose nature can be very complicated.

As an application of D'Alembert's principle let us again focus attention on the plane pendulum problem. The generalized coordinate is chosen as θ. So the work done by the applied force mg is $(-mg \sin \theta)(l\,\delta\,\theta)$ where $\delta\theta$ is the virtual displacement that θ undergoes. Also the acceleration of the bob is $l\ddot{\theta}$. We thus have from (II.19)

$$(-mg)(l\delta\theta \sin \theta) - ml\ddot{\theta}(l\delta\theta) = 0 \qquad \text{(II.20)}$$

or $\ddot{\theta} = -\left(\frac{g}{l}\right)\sin\theta \approx -\frac{g}{l}\theta$

for small θ. This is the usual equation of the simple harmonic motion.

Next suppose that the length of the string is changing with time. Here the only change from (II. 20) is that the pendulum bob has a component of acceleration $\left(l\ddot{\theta} + 2\dot{l}\dot{\theta}\right)$ in the θ-direction. Hence the work done by the inertial force is $-m\left(l\ddot{\theta} + 2\dot{l}\dot{\theta}\right)l\delta\theta$. So (II.20) is modified to the form

$$(-mg)(l\delta\theta \sin \theta) - m\left(l\ddot{\theta} + 2\dot{l}\dot{\theta}\right)l\delta\theta = 0$$

yielding

$$\frac{d}{dt}\left(ml^2\dot{\theta}\right) = -mgl\sin\theta \qquad \text{(II.21)}$$

The lhs of (II.21) is the rate of change of the angular momentum of the bob about the point of support which, in the absence of gravity, remains constant even when the length of the pendulum is changing with time.

Unit : III ❑ Lagrangian Mechanics

III.1 Lagrange's equations of motion : A close inspection of Newton's second law reveals that the number of available equations are fewer than the number of unknowns, the constraint forces being not known beforehand. In this unit, we are going to set up Lagrange's equations of motion (or the generalized equations of motion) a great advantage of which is that the number of unknowns equals the number of degrees of freedom. This is achieved by invoking D'Alembert's principle so that constraint forces are automatically dispensed with. Lagrangian approach opens up a new procedure of handling particle dynamics : the main difference with its Newtonian counterpart lies in the fact that the energies of the system are addressed to rather than the forces themselves.

Recall D'Alembert's principle (II.19) which states that in a virtual displacement $\delta \vec{r}_i$ $(i = 1, 2, \ldots N)$ the work done by the applied forces

$$\delta W^{(a)} = \sum_{i=1}^{N} \vec{F}_i^a . \delta \vec{r}_i \tag{III.2}$$

exactly balance the work done by the inertial forces

$$\delta W^{(in)} = \sum_{i=1}^{N} \left(-m_i \ddot{\vec{r}}_i \right) \cdot \delta \vec{r}_i \tag{III.3}$$

Consider a holonomic N-particle system possessing n degrees of freedom. Then the 3N particle cordinates can be expressed in terms of n generalized coordinates $q_j (j = 1, 2, \ldots n)$

$$\vec{r}_i = \vec{r}_i \; (q_1, \; q_2, \; \ldots q_n; \; t) \tag{III.4}$$

where i = 1, 2, ... N. In a virtual displacement time is to be treated as 'frozen'. As a result the virtual displacement $\delta \vec{r}_i$ has the following variation

$$\delta \vec{r}_i = \sum_{j=1}^{N} \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j, \quad i = 1, 2, \ldots n \tag{III.5}$$

Plugging (III.5) into (III. 2) $\delta W^{(a)}$ acquires the form

$$\delta W^{(a)} = \sum_{j=1}^{n} Q_j . \delta q_j \tag{III.6}$$

where

$$Q_j = \sum_{i=1}^{N} \vec{F}_i^{(a)} . \frac{\partial \vec{r_i}}{\partial q_j} \qquad \text{(III.7)}$$

(III.2) and (III.6) are two equivlent versions of $\delta W^{(a)}$. Since in the rhs of (III.6), Q_j appears attached to δq_j which are virtual displacements for generalized co-ordinates, Q_j may be defined to be the 'generalized force'. However, unlike the vectorial character of the conventional force, it is scalar in character.

If the system under consideration is conservative, $\vec{F}_i^{\,a}$ may be expressed in terms of a potential function i.e. $\vec{F}_i^{\,a} = - \vec{\nabla}_i V$ and $V = V\left(\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N; t\right)$. Consequently (III.7) implies

$$Q_j = - \frac{\partial V}{\partial q_j} \qquad \text{(III.8)}$$

In other words Q_j's one also derivable from the same potential function : $V = V(q_1, q_2, \ldots q_n ; t)$

We now turn to $\delta W^{(\text{in})}$ given by (III. 3). We see that it can be written as

$$\delta W^{(\text{in})} = \sum_{j=1}^{n} \sum_{i=1}^{N} \left(- m_i \ddot{\vec{r}_i} \right) \cdot \frac{\partial \vec{r_i}}{\partial q_j} \delta q_j \qquad \text{(III.9)}$$

on using (III.5). To handle the rhs of (III.9) more effectively we proceed as follows. First we prove two lemmas :

Lemma 1 : If T be the kinetic energy given by (II. 7) then

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_{i=1}^{N} m_i \dot{\vec{r}_i} \cdot \frac{\partial \vec{r_i}}{\partial q_j}, \quad j = 1, 2, \ldots n \qquad \text{(III. 10)}$$

Proof : It is straightforward to deduce from (II. 8) that

$$\frac{\partial \dot{\vec{r}_i}}{\partial \dot{q}_j} = \frac{\partial \vec{r_i}}{\partial q_j} \qquad \text{(III. 11)}$$

Moreover,

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_{i=1}^{N} m_i \dot{\vec{r}_i} \cdot \frac{\partial \dot{\vec{r}_i}}{\partial \dot{q}_j} \qquad \text{(III. 12)}$$

Combining (III. 11) and (III. 12), (III. 10) follows.

Lemma 2 : The operators $\frac{d}{dt}$ and $\frac{\partial}{\partial q_j}$ are commutative in the sense

$$\frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\left(\frac{d\vec{r_i}}{dt}\right) \tag{III. 13}$$

Proof : Lhs of (III. 13) is

$$= \sum_{m=1}^{n} \frac{\partial^2 \vec{r_i}}{\partial q_m \partial q_j} \dot{q}_m + \frac{\partial^2 \vec{r_i}}{\partial t\, \partial q_j}$$

$$= \frac{\partial}{\partial q_j}\left[\sum_{m=1}^{n} \frac{\partial \vec{r_i}}{\partial q_m} \dot{q}_m + \frac{\partial \vec{r_i}}{\partial t}\right]$$

$$= \frac{\partial}{\partial q_j}\left(\frac{d\vec{r_i}}{dt}\right) = \text{Rhs of (III. 13)}$$

Next we combine (III. 10) and (III. 11) to get

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) &= \sum_{i=1}^{N}\left[m_i \ddot{\vec{r_i}} \cdot \frac{\partial \vec{r_i}}{\partial q_j} + m_i \dot{\vec{r_i}} \cdot \frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_j}\right)\right] \\
&= \sum_{i=1}^{N}\left[m_i \ddot{\vec{r_i}} \cdot \frac{\partial \vec{r_i}}{\partial q_j} + m_i \dot{\vec{r_i}} \cdot \frac{\partial}{\partial q_j}\left(\dot{\vec{r_i}}\right)\right] \\
&= \sum_{i=1}^{N} m_i \ddot{\vec{r_i}} \cdot \frac{\partial \vec{r_i}}{\partial q_j} + \frac{\partial}{\partial q_j}\left[\sum_{i=1}^{N} \frac{1}{2} m_i \left|\dot{\vec{r_i}}\right|^2\right] \\
&= \sum_{i=1}^{N} m_i \ddot{\vec{r_i}} \cdot \frac{\partial \vec{r_i}}{\partial q_j} + \frac{\partial T}{\partial q_j}
\end{aligned} \tag{III. 14}$$

where we have also used (III. 13).

Finally we substitute (III.14) in (III.9) to arive at

$$\delta W^{(in)} = \sum_{j=1}^{n}\left[\frac{\partial T}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial t}{\partial \dot{q}_j}\right)\right]\delta q_j \tag{III. 15}$$

The representations (III. 6) and (III. 15) enable us to restate D'Alembert's principle (II. 19) in the following form :

$$\sum_{j=1}^{n}\left[Q_j + \frac{\partial T}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial t}{\partial \dot{q}_j}\right)\right]\delta q_j = 0 \qquad \text{(III. 16)}$$

The quantities δq_j being arbitrary and independent, it transpires from (III. 16) that the coefficients of the each δq_j must vanish separately.

In consequence it must be true that

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j, \quad j = 1, 2, \ldots n \qquad \text{(III.17)}$$

A moment's throught would revel that (III. 17) actually describes a set of n second-order differential equations involving n generalized quantities and their velocities.

For a conservative system when (III. 8) holds, (III. 17) can be expressed in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad \text{(III.18)}$$

where $L = T - V$ and V is indepdnent of the velocities $\dot{q}_1, \dot{q}_2, \ldots \dot{q}_n$. L is called the Lagrangian and (III. 17), (III. 18) are referred to as Lagrange's equations of motion.

It is to be stressed that the unknown forces of constraint are absent from the Lagrange's equations. Further, unlike Newton's equations of motion, there is no direct reference to quantities like force or acceleration. In contrast, only a knowledge of kinetic and potential enerties are required to set up (III. 18). However, the appearance of the generalized force Q_j in the rhs of Eq (III. 17), which in turn is related to the applied forces $\vec{F}_i^{\,a}$ as given by (III.7), signals that the kinetic energy T needs to be evaluated in an inertial frame. The reason is that $\vec{F}_i^{\,a}$ have their origins in Newton's Laws which are valid in inertial frames only.

The form (III. 18) admits of the addition of a total derviative term to L without affecting the equations of motion. For if we construct a new Lagrangian L' from L according to

$$L'(q_j, \dot{q}_j, t) = L(q_j, \dot{q}_j, t) + \frac{d\lambda}{dt}, j = 1, 2, \ldots n$$

where λ is any differentiable function of position and time, then it is trivial to check that $\frac{d\lambda}{dt}$ gives a vanishing contribution to the equations of motion :

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\frac{d\lambda}{dt}\right)\right] = \frac{\partial}{\partial q_j}\left(\frac{d\lambda}{dt}\right) \qquad \text{(III. 19)}$$

To prove (III. 19) we simply have to note that $\frac{d\lambda}{dt} = \sum_{j=1}^{n} \frac{\partial \lambda}{\partial q_j}\dot{q}_j + \frac{\partial \lambda}{\partial t}$ and the result immediately follows. We therfore conclude that both L and L' lead to the same equations of motion.

Sometimes, depending upon the nature of the problem, a more general representation of Q_j than the one given in (III. 8) is called for. Suppose a velocity dependent potential exists namely $U(q_j, \dot{q}_j, t)$ such that Q_j is derivable from it in the manner

$$Q_j = -\left[\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) - \frac{\partial U}{\partial q_j}\right], \; j = 1, 2, \dots n \qquad \text{(III. 20)}$$

then a similar set of equations as (III. 18) follows from (III. 17) if L is defined according to $L = T - U$. In unit IV we shall see that a velocity dependent potential has relevance in setting up of a Lagrangian for rotating frames.

III. 2 Lagrange's equations for some simple systems :

(i) Plane pendulum : Referred to Fig. II.1, if θ be the generalized coordinate, then the kinetic and poterrtial energies are

$$T = \frac{1}{2}ml^2\dot{\theta}^2, \; V = -mgl\cos\theta$$

As a result the Lagrangian is

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

It yields

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

From (III. 18), the equation of motion turns out to be

$$ml^2\ddot{\theta} = -mgl\sin\theta$$

For small θ, it can be replaced by

$$\ddot{\theta} = -\frac{g}{l}\theta$$

Note that while $\frac{\partial L}{\partial \dot{\theta}}$ stands for the angular momentum of the mass about the point of support, $\frac{\partial L}{\partial \theta}$ represents the torque.

Remark : θ is not the only generalized coordinate we can employ. However, as it happens, any other choice leads to a more complex form of the equation of motion. For instance, if we use the horizontal displacement $x(= l\sin\theta)$ as the generalized coordinate, then the forms of T and V are $T = \frac{1}{2}m\frac{l^2\dot{x}^2}{l^2 - x^2}$ and $V = -mg\sqrt{l^2 - x^2}$. With $L = T - V$, the equation of motion reads $\ddot{x} = -\frac{x\dot{x}^2}{l^2 - x^2} - \frac{gx}{l^2}\sqrt{l^2 - x^2}$ which is more complicated to solve than the one for θ.

(ii) Spherical pendulum : Here the bob of the pendulum can swing in any direction. As a result m traces out a sphere of constant length l. Using the polar co-ordinates θ and ϕ and noting that in spherical polar co-ordinates the form for the kinetic energy is $T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right)$, we find for the present problem

$$T = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)$$

$$V = -mg\,l\cos\theta$$

where V is the same as in the place pendulum problem.

Here the number of generalized coordinates is two, θ and ϕ.

The Lagrangian $L = T - V$ gives

$$\frac{\partial L}{\partial \theta} = ml^2\sin\theta\cos\theta\,\dot{\phi}^2 - mg\,l\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{\partial L}{\partial \phi} = 0$$

$$\frac{\partial L}{\partial \dot{\phi}} = ml^2 \sin^2 \theta \, \dot{\phi}$$

The equations of motion are

$$\ddot{\theta} = \sin \theta \cos \theta \, \dot{\phi}^2 - \frac{g}{l} \sin \theta$$

$$\frac{d}{dt}\left(ml^2 \sin^2 \theta \, \dot{\phi}\right) = 0$$

The second equation implies that the angular momentum in the vertical direction is a constant of motion. We have $\dot{\phi} = \frac{A}{ml^2 \sin^2 \theta}$, A is a constant. The θ-equation of motion turns out to be

$$\dot{\theta}^2 = -\frac{A^2}{m^2 l^2} \cot^2 \theta + \frac{2g}{l} \cos \theta + B$$

where B is a constant.

(iii) Binary star system : Consider a binary star consisting of two masses m and M. If O denotes the fixed origin and N the centre of mass then according to the figure III. 1 :

$$\vec{r}_1 = \vec{R} + \vec{r}_1{}', \quad \vec{r}_2 = \vec{R} + \vec{r}_2{}'$$

where $\vec{R}$ is the centre of mass :

$$\vec{R} = \frac{m\vec{r}_1 + M\vec{r}_2}{m + M}.$$

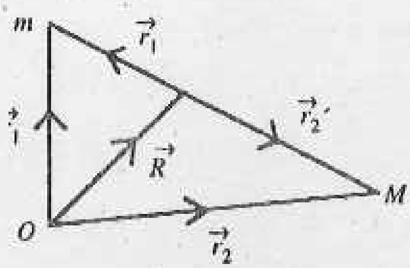


Fig. III. 1

The kinetic and potential energies are

$$T = \frac{1}{2} m \left|\dot{\vec{r}}_1\right|^2 + \frac{1}{2} M \left|\dot{\vec{r}}_2\right|$$

$$V = -\frac{GmM}{\left|\vec{r}_1 - \vec{r}_2\right|}$$

The Lagrangian can be written in the form

$$L = \frac{1}{2}\left[m\left|\dot{\vec{r}}_1\right|^2 + M\left|\dot{\vec{r}}_2\right|^2\right] + \frac{GmM}{\left|\vec{r}_1 - \vec{r}_2\right|}$$

$$= \frac{1}{2}(m+M)\dot{\vec{R}}^2 + \frac{1}{2}\frac{mM}{m+M}\left|\dot{\vec{r}}\right|^2 + \frac{GmM}{\left|\vec{r}\right|}$$

where $\vec{r} = \vec{r}_1 - \vec{r}_2 = \vec{r}_1{}' - \vec{r}_2{}'$. We find as a consequence $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{R}}}\right) = 0$, the coordinate $\vec{R}$ not appearing explicitly in L. So $(m+M)\dot{\vec{R}}$ = constant implying that the kinetic energy of the system as a whole (see the first term in the Lagrangian) remains constant. This term is of no consequence while studying the internal motion of the system.

We now solve some problems.

III.3 Some solved problems :

1. For a system having four degrees of freedom the Lagrangian is given by

$$L = m\left[\dot{q}_4^2 - \dot{q}_1^2 - \dot{q}_2^2 - \dot{q}_2^3\right]^{1/2} + e\sum_{\mu=1}^{4} A_\mu \dot{q}_\mu$$

where A_μ's are functions of coordinates alone and m, e are constants.

Show that the equations of motion can be put in the form

$$m\frac{d}{dt}\left(\lambda \dot{q}_j\right) = e\sum_{k=1}^{4}\left(\frac{\partial A_j}{\partial q_k} - \frac{\partial A_k}{\partial q_j}\right)\dot{q}_k,\ j = 1, 2, 3$$

$$m\frac{d}{dt}\left(\lambda \dot{q}_4\right) = e\sum_{k=1}^{4}\left(\frac{\partial A_k}{\partial q_4} - \frac{\partial A_4}{\partial q_k}\right)\dot{q}_k$$

where $\lambda^2 = \left(\dot{q}_4^2 - \dot{q}_1^2 - \dot{q}_2^2 - \dot{q}_3^2\right)^{-1}$

Answer : For the coordinates $q_j, j = 1, 2, 3$ we have

$$\frac{\partial L}{\partial \dot{q}_j} = -m\dot{q}_j\left(\dot{q}_4^2 - \dot{q}_1^2 - \dot{q}_2^2 - \dot{q}_2^3\right)^{-1/2} + eA_j$$

$$= -m\lambda\, \dot{q}_j + eA_j$$

$$\frac{\partial L}{\partial q_j} = e\sum_{\mu=1}^{4}\frac{\partial A_\mu}{\partial q_j}\dot{q}_\mu$$

Using Lagrange's equation of motion we get

$$\frac{d}{dt}\left(-m\lambda\dot{q}_j + eA_j\right) = e\sum_{\mu=1}^{4}\frac{\partial A_\mu}{\partial q_j}\dot{q}_\mu$$

Writing $\frac{dA_j}{dt} = \sum_{k=1}^{4}\frac{\partial A_j}{\partial q_k}\dot{q}_k$, A_j being functions of position only, the result follows.

For q_4, we have

$$\frac{\partial L}{\partial \dot{q}_4} = m\,\lambda\,\dot{q}_4 + eA_4$$

$$\frac{\partial L}{\partial q_4} = \sum_{\mu=1}^{4}\frac{\partial A_\mu}{\partial q_4}\dot{q}_\mu$$

Writing $\frac{dA_4}{dt} = \sum_{\mu=1}^{4}\frac{\partial A_4}{\partial q_\mu}\dot{q}_\mu$ and using Lagrange's equation of motion for q_4 the result follows.

2. Study the problem of damped oscillator described by the Lagrangian

$$L = e^{\gamma t}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2\right)$$

where γ, m, k are positive constants by considering the different cases. Justify the choice of the Lagrangian.

Answer : For the Lagrangian the equation of motion is

$$\ddot{q} + \gamma\dot{q} + \frac{k}{m}q = 0 \quad \text{(damped oscillator)} \quad \text{(i)}$$

Trying a solution of the type $q \sim e^{\alpha t}$ from the linearlity of the equation, we find that the constant α is resticted by

$$\alpha = -\frac{\gamma}{2} \pm b,\ b = \sqrt{\left(\frac{\gamma}{2}\right)^2 - \frac{k}{m}}$$

There cases may arise

(i) $\frac{\gamma}{2} < \sqrt{\frac{k}{m}}$: $q = e^{-\frac{\gamma t}{2}}$

where A, B are constants and $\beta = -ib$ so that β is real.

(ii) $\frac{\gamma}{2} = \sqrt{\frac{k}{m}}$: $q = q_0 e^{-\frac{\gamma t}{2}}$ where $q = q_0$ at $t = 0$

(iii) $\frac{\gamma}{2} > \sqrt{\frac{k}{m}}$: $q = e^{-\frac{\gamma t}{2}} \left(Ce^{bt} + De^{-bt}\right)$

where C, D are constants and b is real. Since $b > \frac{\gamma}{2} > 0$,

we have to set $C = 0$ to avoid an explosive term.

It is easy to be observed that should we set $q = e^{-\frac{\gamma t}{2}} s$ then $\dot{q} = \left(\dot{s} - \frac{s\gamma}{2}\right) e^{-\frac{\gamma t}{2}}$ and $\ddot{q} = \left(\ddot{s} - \dot{s}\gamma + \frac{s\gamma^2}{4}\right) e^{-\frac{\gamma t}{2}}$ and $\ddot{q} = \left(\ddot{s} - \dot{s}\gamma + \frac{s\gamma^2}{4}\right) e^{-\frac{\gamma t}{2}}$. So (i) is transformed to $\ddot{s} + \omega^2 s = 0$ where $\omega^2 = \frac{k}{m} - \frac{\gamma^2}{4}$. It resembles the form of a simple harmonic motion for $\frac{\gamma}{2} < \sqrt{\frac{k}{m}}$.

Note that it is not possible to obtain the form (i) for the equation of the damped oscillator simply by adding to the Lagarangian of the simple harmonic motion, namely $L_{SHM} = \frac{1}{2} m\dot{q}^2 - \frac{1}{2} m\omega^2 q^2$, a potential term corresponding to the damping force $(-\gamma m\dot{q})$ i.e. $-\gamma mq\dot{q}$. The resulting Lagrangian $L' = \frac{1}{2} m\dot{q}^2 - \frac{1}{2} m\omega^2 q^2 - \gamma mq\dot{q}$ yields $\ddot{q} + \omega^2 q = 0$ instead of (i). The additional team $-\gamma mq\dot{q}$ is actually a total derivative $\frac{d\wedge}{dt}$, $\wedge = -\frac{1}{2}\gamma mq^2$. It is therefore not ssurprising that both L_{SHM} and L' lead to the same equation of motion. The correct Lagrangian for the damped oscillator must contain an explicit time dependence in the form of an overall exponential factor as given in our problem.

3. Show that for a conservative, scleronomic system the quantity $\sum_{j=1}^{n} \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$ is a constant and that if the kinetic energy is a homogeneous quadratic function of velocities, the total energy is constant as well.

Answer : Since the system is conservative and scleronomic,

$$\frac{d}{dt}\left[\sum_{j=1}^{n} \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L\right] = \sum_{j=1}^{n}\left[\ddot{q}_j \frac{\partial L}{\partial \dot{q}_j} + \dot{q}_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right]$$

$$- \sum_{j=1}^{n}\left[\frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial L}{\partial q_j}\dot{q}_j\right]$$

$$= 0$$

where we have exploited Lagranges' equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j}$ and noted that for a scleronomic system $\frac{\partial L}{\partial t} = 0$

$$\therefore \quad \sum_{j=1}^{n} \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = \text{constant}$$

Next,

$$\sum_{j=1}^{n} \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = \sum_{j=1}^{n} \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} - (T - V)$$

$$\left(\because \frac{\partial V}{\partial \dot{q}_j} = 0 \text{ for a consevative system}\right)$$

$$= 2T - (T - V) \quad \text{(by Euler's theorem)}$$

$$= T + V$$

Combining this with the previous result we arrive at the conclusion that $T + V$ = total energy = constant.

III. 4 Central force and orbits : As already noted in Unit II, the central force is a two-dimensional problem. In plane polar co-ordinates (r, θ) the Lagrangian is given by

$$\text{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r) \qquad \text{(III. 21)}$$

The resulting equations of motion are

$$m\ddot{r} = mr\dot{\theta}^2 - V'(r) \tag{III. 22}$$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0 \tag{III. 23}$$

where $V'(r) \equiv \frac{dV}{dr}$. The second equation is a statement of the constancy of areal velocity $\frac{1}{2}r^2\dot{\theta}$ and also points to the conservation of the angular momentum $mr^2\dot{\theta}$.

From Fig. III.2 it actually follows that the radius vector sweeps out, in time dt, a differential area $dA = \frac{1}{2}(r)(r\,d\theta) = \frac{1}{2}(r)\left(r\dot{\theta}\right)dt$.

In other words

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \text{constant} = \frac{l}{2m}, \text{ say} \tag{III. 24}$$

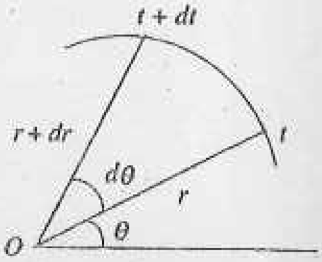


Fig. III. 2

Thus the motion of the particle along its orbit is such that the radius vector sweeps out equal areas in equal times. This is Kepler's second law. Note that we have not used any specific form for $V(r)$ so the property of the constancy of areal velocity is true for a general central force problem.

Integrating (III. 22) and using (III.23) and (III. 24)

we get

$$m\ddot{r} = -\frac{dU(l,r)}{dr} \tag{III. 25}$$

where $U(l, r) = \frac{l^2}{2mr^2} + V(r)$ noted earlier in (I.5). Recall that $\frac{l^2}{2mr^2}$ is the centrifugal term V_{cf}. For the Newtonian inverse square law, $V(r) = -\frac{\mu}{r}$ (which is negative for an attractive force, $\mu > 0$), a graphical description of $U(l, r)$ and V_{cf} is illustrated in Fig III.3. We see that while V_{cf} goes as $\frac{1}{r^2}$, $V(r)$ is always negative. As a result, $U(l, r)$ can descend to a minimum having a finite negative value.

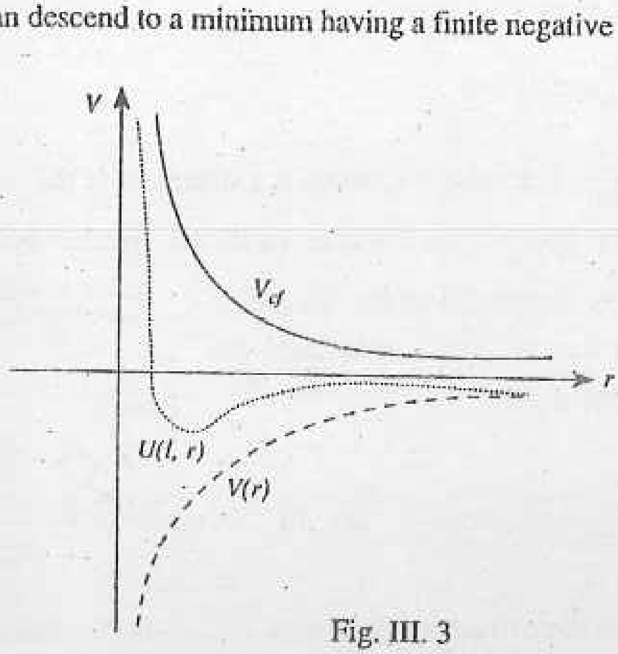


Fig. III. 3

Not only the inverse but the entire class of power-law potentials can be described by the form

$$V(r) = -\frac{\gamma}{r^{\nu}} = -\frac{\gamma}{r^{2\lambda+2}} \qquad \text{(III.26)}$$

where $\lambda = \frac{1}{2}(\nu - 2)$. The above potential are attractive for $\nu > 0$ but repulsive for $\nu < 0$. Corresponding to (III. 26) we can define

$$U_p(l, r) = \frac{l^2}{2mr^2} - \frac{\gamma}{r^{\nu}} \qquad \text{(III. 27)}$$

i.e. $$U_p(\rho) = \frac{l^2}{2ma^2}\left(\frac{1}{\rho^2} - \frac{1}{\rho^{\nu}}\right) \qquad \text{(III. 28)}$$

where $\rho = \frac{r}{a}$. If a is determined from the condition $U_p(l, a) = 0$ then a turns out to be

$$U_p(l,\ a) = 0\ :\ a = \left(\frac{2m\gamma}{l^2}\right)^{1/(\nu-2)} \qquad \text{(III. 29)}$$

Further $V(r)$ assumes the form

$$V(r) = -\frac{l^2 a^{\nu-2}}{2m}\frac{1}{r^{\nu}} \tag{III. 30}$$

We speak of a bounded motion if $r = r_{min}$ and $r = r_{max}$ exist where $\dot{r} = 0$. From (I.6), obviously $E = U(l, r) = \frac{1}{2} m\dot{r}^2 > 0$ for all r, we have the condition $U(l, r) \leq E$ for a physical motion. In Fig III.3, the curve, $U(l, r)$ reaches a minimum with a finite negative value implying a range of bounded orbits.

We now make some remarks on the circular orbits which are, from the mathematical point of view, the simplest ones to handle. It is clear from (I.3) that there will be a circular orbit of radius R if

$$F(R) + mR\dot{\theta}^2 = 0$$

i.e. $$F(R) + \frac{l^2}{mR^3} = 0 \tag{III. 31}$$

This also follows from $\left.\frac{d\,U(l,r)}{dr}\right|_{r=R} = 0$ and putting $F = -\frac{dV}{dr}$. (III. 31) is just a re-interpretation of the criterion that a particle in a circular orbit has a constant acceleration v^2/R towards the centre and so $\frac{mv^2}{R} = -F(R)$ $[F(R) < 0, v = R\dot{\theta}]$.

Writing (III. 25) as

$$m\ddot{r} = m\frac{d^2}{dt^2}(r - R) = -U'(l,r) \tag{III.32}$$

where $U'(l,r) = \frac{dU(l,r)}{dr}$, we expand the rhs as

$$U'(l,r) \approx U'(l,R) + (r - R)U''(l,R)$$

$$= -(r - R)\,U''(l,R)$$

Thus

$$m\frac{d^2}{dt^2}(r - R) \approx -(r - R)U''(l,R) \tag{III.33}$$

and we conclude that the circular orbit would be stable if $U''(l,R) > 0$. Eq. (III.33) has the form $\ddot{x} = -\omega^2 x$. For a positive $U''(l,R)$ we can hence estimate the angular frequency of small oscillations about a stable circular orbit :

$$\omega = \sqrt{\frac{U''(l,R)}{m}} = \sqrt{\frac{v''(R) + 3l^2/mR^4}{m}} \qquad \text{(III. 34)}$$

Because of (III.31), we can eliminate l to derive

$$\omega = \sqrt{\frac{1}{m}\left[V''(R) + \frac{3V'(R)}{R}\right]} \qquad \text{(III.35)}$$

The period is $T = 2\pi/\omega$.

Problem : Find the stability of circular orbits for the power law potentials

$$v(r) = -\frac{\gamma}{r^{\nu}},\ \nu \neq 0,\ \gamma \neq 0$$

Answer : The force corresponding to $V(r)$ must be attractive i.e.

$$F = -\frac{dV}{dr} < 0.$$

It gives $-\nu\gamma r^{-\nu-1} < 0$. So we must have $\nu\gamma > 0$.

The condition of stability is $U''(l,R) > 0$ i.e. $V''(R) + \frac{3V'(R)}{R} > 0$

[see (III.35)]. we get

$$-\gamma\nu(\nu+1)R^{-\nu-2} + \frac{3}{4}(\gamma\nu)R^{-\nu-1} > 0$$

or, $-\gamma\nu[(\nu+1) - 3] > 0$

or, $-\gamma\nu(\nu - 2) > 0$

Since $\gamma\nu > 0$ we are led to $\nu < 2$.

III.5. Ignorable co-ordinates : It often happens that for a system with n degrees of freedom, a set of generalized coordinates, say $q_1, q_2, \ldots, q_k$, are not explicitly present in the Lagrangian althengh L contains the corresponding velocities $\dot{q}_1, \dot{q}_2, \ldots \dot{q}_k$. Such coordinates are then called ignorable (or cyclic) coordinates. We have already encountered ignorable coordinates in some of the problems we have come across : for example, in the central force problem θ is the ignorable coordinate while in the spherical pendulum ϕ is absent fum the Lagrangian and as such it is ignorable. It should be clear that corresponding to any ignorable coordinate, Lagrange's equation reveals an associated constant of motion. Out task here would be to set up a modified Lagrangian addressing to the remaining coordinates $q_{k+1}, q_{k+2}, \dots q_n$ from which the equations of motion can be derived. Such a modified Lagrangian is called the Routhian.

We first of all observe that for the ignorable coordinates $q_1, q_2, \dots q_k$, Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) = 0, \quad r = 1, 2, \dots k$$

$$\text{i.e.} \quad \frac{\partial L}{\partial \dot{q}_r} = \beta_r, \quad r = 1, 2, \dots k \tag{III.36}$$

where $\beta_1, \beta_2, \dots \beta_k$ are the constants of intergration.

Define now the quantity

$$R = L - \sum_{r=1}^{k} \dot{q}_r \frac{\partial L}{\partial \dot{q}_r} \tag{III.37}$$

called the Routhian. Since because of (III.36) we can always express $\dot{q}_1, \dot{q}_2, \dots \dot{q}_k$ in terms of the coordinates $q_{k+1}, q_{k+2}, \dots q_n$, the velocities $\dot{q}_{k+1}, \dot{q}_{k+2}, \dots, \dot{q}_n$ and the quantities $\beta_1, \beta_2, \beta_k$, we can write R as

$$R = R[q_{k+1}, q_{k+2}, \dots q_n ; \dot{q}_{k+1}, \dot{q}_{k+2}, \dots \dot{q}_n ; \beta_1, \beta_2, \dots \beta_k] \tag{III.38}$$

To proceed further we calculate the increment of both sides of (III.37) i.e.,

$$\delta R = \delta\left[L - \sum_{r=1}^{k} \dot{q}_r \frac{\partial L}{\partial \dot{q}_r}\right] \tag{III.39}$$

Now L being $L = L(q_{k+1}, q_{k+2}, \dots q_n ; \dot{q}_1, \dot{q}_2, \dots \dot{q}_k, \dots \dot{q}_n ; t)$ *it* gives

$$\delta L = \sum_{r=k+1}^{n} \frac{\partial K}{\partial q_r} \delta q_r + \sum_{r=1}^{n} \frac{\partial L}{\partial \dot{q}_r} \delta \dot{q}_r + \sum_{r=k+1}^{n} \frac{\partial L}{\partial \dot{q}_r} \delta \dot{q}_r$$

where t is treated as forezen because of the virtual variation. Also,

$$\delta\left[\sum_{r=1}^{k} \dot{q}_r \frac{\partial L}{\partial \dot{q}_r}\right] = \sum_{r=1}^{k} \delta \dot{q}_r \frac{\partial L}{\partial \dot{q}_r} + \sum_{r=1}^{k} \dot{q}_r \delta \beta_r$$

So (III.39) becomes

$$\delta \mathrm{R} = \sum_{r=k+1}^{n} \frac{\partial L}{\partial q_r} \delta q_r + \sum_{r=k+1}^{n} \frac{\partial L}{\partial \dot{q}_r} \delta \dot{q}_r - \sum_{r=1}^{k} \dot{q}_r \delta \beta_r \tag{III.40}$$

Now δR itself is because of (III.38)

$$\delta R = \sum_{r=k+1}^{n} \frac{\partial R}{\partial q_r} \delta q_r + \sum_{r=k+1}^{n} \frac{\partial R}{\partial \dot{q}_r} \delta \dot{q}_r + \sum_{r=1}^{k} \frac{\partial R}{\partial \beta_r} \delta \beta_r \tag{III.41}$$

Comparing (III.40) and (III.41) and noting that the variations are arbitrary and independent we deduce

$$\begin{aligned} \frac{\partial L}{\partial q_r} &= \frac{\partial R}{\partial q_r}, \ r = k+1, k+2, \dots n \\ \frac{\partial L}{\partial \dot{q}_r} &= \frac{\partial R}{\partial \dot{q}_r}, \ r = k+1, k+2, \dots n \\ \dot{q}_r &= -\frac{\partial R}{\partial \beta_r}, \ r = 1, 2, \dots k \end{aligned} \tag{III.42}$$

The first two equations of (III.42) give

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_r}\right) = \frac{\partial R}{\partial q_r}, \ r = k+1, k+2, \dots n \tag{III.43}$$

while the last one implies

$$q_r = \int \frac{\partial R}{\partial \beta_r} dt, \ r = 1, 2, \dots k \tag{III.44}$$

The message of the set of equations (III.43) is that, knowing R, $q_{k+1}, q_{k+2}, \dots q_n$ can be determined in terms of t. Having got this, the remaining coordinates are obtained from (III.44).

Ex. 1. In a dynamical system the kinetic and potential energies are

$$T = \frac{1}{2}\frac{\dot{q}_1^2}{a + bq_2^2} + \frac{1}{2}\dot{q}_2^2, \quad V = c + dq_2^2$$

Determine $q_1(t)$ and $q_2(t)$ by Routh's process of ignoration of coordinates.

Answer :

The Lagrangian is

$$L = \frac{1}{2}\frac{\dot{q}_1^2}{a + bq_2^2} + \frac{1}{2}\dot{q}_2^2 - c - dq_2^2$$

$$\Rightarrow \quad \frac{\partial L}{\partial \dot{q}_1} = \frac{\dot{q}_1}{a + bq_2^2} = \beta$$

where q_1 is the ignorable coordinate. The Routhian is

$$R = L - \dot{q}_1 \frac{\partial L}{\partial \dot{q}_1}$$

which can be written as

$$R = \frac{1}{2}\dot{q}_2^2 - \left(d + \frac{1}{2}b\beta^2\right)q_2^2 - c - \frac{1}{2}a\beta^2$$

involving q_2 and $\dot{q}_2$ only.

Equation (III.43) gives

$$\ddot{q}_2 + \left(2d + b\beta^2\right)q_2 = 0$$

Solving,

$$q_2 = A\sin\left[\left(2d + b\beta^2\right)^{1/2} t + \epsilon\right]$$

where A and ϵ are constants of integration. The coordinate q_1 is obtained from (III.44) :

$$q_1 = -\int \frac{\partial R}{\partial \beta}\, dt$$

$$= \beta \int \left(a + bq_2^2\right) dt$$

Substituting the above solution of q_2 we get

$$q_1 = \left(\beta a + \frac{1}{2}\beta b A^2\right)t - \frac{\beta b A^2}{4(2d + b\beta^2)^{1/2}} \sin\left[\left(2d + b\beta^2\right)^{1/2} t + \epsilon\right] + B$$

where B is a constant of integration.

Ex. 2. Solve the planetary problem by Routh's process of ignoration of co-ordinates.

Answer :

Here $T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$

$V = -\frac{\mu}{r}$

$L = T - V = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{\mu}{r}$

Since L does not contain θ, it is an ignorable co-ordinate :

$$\frac{\partial L}{\partial \dot{\theta}} = \text{constant} = l \text{ (say)}$$

$$\text{or, } ml^2\dot{\theta} = l$$

This equation expresses the conservation of angular momentum. We now set up the Routhan

$$R = L - \dot{\theta}\frac{\partial L}{\partial \dot{\theta}}$$

$$= \frac{m}{2}\left[\dot{r}^2 - r^2\frac{l^2}{m^2r^4}\right] + \frac{\mu}{r}$$

Note that R contains r and $\dot{r}$ only. We then get from (III.43)

$$m\ddot{r} = \frac{l^2}{mr^3} - \frac{\mu}{r^2}$$

Its first integral gives the conservation of total energy E :

$$\frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{l^2}{mr^2} + V = \text{constant} = \text{E} \quad \ldots \quad \text{(i)}$$

as we have laready seen in connection with the central force problem.

On the other hand, Eq. (III.44) gives

$$\theta = -\int\frac{\partial R}{\partial l}dt$$

$$= l\int_0^t \frac{dt}{mr^2(t)} + \theta_0 \quad \ldots \quad \text{(ii)}$$

where θ_0 is the initial value of θ. Note that (i) can be written in the form

$$t = \int_{r_0}^{r}\frac{dr}{\left[\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)\right]^{1/2}} \quad \ldots \quad \text{(iii)}$$

where r_0 is the initial value of r. From (i), (ii), (iii) we can indentify four constants of integration l, E, r_0 and θ_0.

Liouville's class of Lagragians : The Liouville's class of Langrangians is the one for which the kinetic and potential energies are in the form

$$T = \frac{1}{2}\left[u_1(q_1) + u_2(q_2) + \ldots + u_n(q_n)\right]\left[v_1(q_1)\dot{q}_1^2 + v_2(q_2)\dot{q}_2^2 + \ldots + v_n(q_n)\dot{q}_n^2\right]$$

$$V=\left[w_1(q_1)+w_2(q_2)+\cdots+w_n(q_n)\right]/\left[u_1(q_1)+u_2(q_2)+\ldots+u_n(q_n)\right]$$

where u_r, v_r, w_r are functions of q_r only, $r = 1, 2, \ldots n$ for a system having n degrees of freedom and the forces are conservative in nature. A great advantage with the problems of Liouville's type is that these admit separation of variables and hence can be solved completely.

Let us set $\dot{Q}_1^2 = v_1(q_1)\dot{q}_1^2$, $\dot{Q}_2^2 = v_2(q_2)\dot{q}_2^2, \ldots, \dot{Q}_n^2 = v_n(q_n)\dot{q}_n^2$

which in turn imply

$$dQ_r = \sqrt{v_r(q_r)}\,dq_r$$

Integrating, Q_r is obtained as a function of q_r alone, $r = 1, 2, \ldots, n$.

As a result we can transform $u_r(q_r)$ to $U_r(Q_r)$ and $w_r(q_r)$ to $W_r(Q_r)$. Thus T and V acquire the forms

$$T=\frac{1}{2}U\sum_{r=1}^{n}\dot{Q}_r^2$$

$$V=\frac{W}{U}$$

where $U=\sum_{r=1}^{n}U_r(Q_r)$ and $W=\sum_{r=1}^{n}W_r(q_r)$.

In terms of Q_r, $r = 1, 2, \ldots, n$, which are the new generalized coordinates, Lagrange's equations read

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}_r}\right)-\frac{\partial T}{\partial Q_r}=-\frac{\partial V}{\partial Q_r}$$

i.e. $$\frac{d}{dt}\left(U\dot{Q}_r\right)-\frac{1}{2}\frac{\partial U}{\partial Q_r}\left(\sum_{r=1}^{n}\dot{Q}_r^2\right)=-\frac{\partial V}{\partial Q_r},\quad r=1,2,\ldots,n$$

Muthplying both sides by $U\dot{Q}_r$ we get

$$\frac{d}{dt}\left(\frac{1}{2}U^2\dot{Q}_r^2\right)-T\,\dot{Q}_r\frac{\partial U}{\partial Q_r}+U\dot{Q}_r\frac{\partial V}{\partial Q_r}=0 \qquad \text{(III.45)}$$

We have already noted that for a conservative, scleronomic system, if the kinetic energy is a homogeneous quadratic function of velocities then the total energy is constant : $T + V = h$, h is a constant.

As such (III.45) can be put in the form

$$\frac{d}{dt}\left(\frac{1}{2}U^2\dot{Q}_r^2\right) - \dot{Q}_r\left[h\frac{\partial U}{\partial Q_r} - \frac{\partial}{\partial Q_r}(UV)\right] = 0$$

i.e. $$\frac{d}{dt}\left(\frac{1}{2}U^2\dot{Q}_r^2\right) - \dot{Q}_r\left[h\frac{\partial U}{\partial Q_r} - \frac{\partial W}{\partial Q_r}\right] = 0$$

or, $$\frac{d}{dt}\left(\frac{1}{2}U^2\dot{Q}_r^2\right) - \dot{Q}_r\left[h\frac{\partial U_r}{\partial Q_r} - \frac{\partial W_r}{\partial Q_r}\right] = 0,\ r = 1, 2, \ldots n$$

Since $\dot{Q}_r\frac{\partial U_r}{\partial Q_r} = \frac{dU_r}{dt}$ and $\dot{Q}_r\frac{\partial W_r}{\partial Q_r} = \frac{dW_r}{dt}$ we arrive at the constants of motion

$$\frac{1}{2}U^2\dot{Q}_r^2 - hU_r + W_r = C_r,\ r = 1, 2, \ldots n \tag{III.46}$$

where C_r's are the constants of integration.

Denoting

$$d_r = C_r + h\,U_r - W_r$$

the equations of motion $\frac{1}{2}U^2\dot{Q}_r^2 = d_r$ translate to

$$\frac{1}{2}u^2\,v_r\,\dot{q}_r^2 = d_r(q_r)$$

in terms of the old variables. In consequence we have

$$\sqrt{\frac{v_r(q_r)}{d_r(q_r)}}\,dq_r = \frac{\sqrt{2}}{u}\,dt,\ r = 1, 2, \ldots n \tag{III.47}$$

More explicilty

$$\int\sqrt{\frac{v_1(q_1)}{d_1(q_1)}}\,dq_1 = \int\sqrt{\frac{v_2(q_2)}{d_2(q_2)}}\,dq_2 + \beta_1 = \ldots = \int\sqrt{\frac{v_n(q_n)}{d_n(q_n)}}\,dq_n + \beta_{n-1} \tag{III.48}$$

In this way the variables become seperated. Multiplying (III. 48) by $u_1, u_2, \ldots u_n$ for each respective value of $r = 1, 2, \ldots n$ and adding we get

$$\sum_{r=1}^{n}\int u_r\sqrt{\frac{v_r}{d_r}}\,dq_r = \int\sqrt{2}\,\frac{\sum u_r}{u}\,dt + c$$

$$= \sqrt{2}t + c \tag{III.49}$$

where c is the constant of integration. (III.48) and (III.49) provide the complete solution to Liouville's problem. Note that the complete solution is subject to $C_1 + C_2 + \dots C_n = 0$. This can be seen very easily from (III.46) which can be written as $\frac{1}{2}u^2 v_r(q_r)\dot{q}_r^2 = C_r - w_r(q_r) + h\, u_r(q_r)$. Summing up we get $\frac{1}{2}u \cdot \frac{u}{2}\sum_{r=1}^{n} v_r(q_r)\dot{q}_r^2 = \sum_{r=1}^{n} C_r - \sum_{r=1}^{n} w_r + h\sum_{r=1}^{n} u_r$. In other words, $\sum C_r = u(T + V - h) = 0$.

Ex. 1. Show that the dynamical system for which $2T = r_1 r_2\left(\dot{r}_1^2 + \dot{r}_2^2\right)$ and $V = \frac{1}{r_1} + \frac{1}{r_2}$ can be expressed as one of Liouville's types.

Answer : Put $r_1 = q_1 + q_2$ and $r_2 = q_1 - q_2$. T and V because

$$T = \left(q_1^2 - q_2^2\right)\left(\dot{q}_1^2 + \dot{q}_2^2\right)$$

$$V = \frac{2q_1}{q_1^2 - q_2^2}$$

and the problem can be recognized to be of Liouville's type.

Ex.2. A particle moves in a plane under the action of two Newtonian centres of attraction at the points $(c, 0)$ and $(-c, 0)$ the attractions being $\frac{\mu}{r^2}$ and $\frac{\mu'}{r'^2}$ respectively ; r, r' being the distances from $(c, 0)$ and $(-c, 0)$ respectively. Show that the problem is of Liouville's type.

Answer : Here

$$T = \frac{1}{2}mv^2 = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right)$$

$$V = -\frac{\mu}{r} - \frac{\mu'}{r'} = -\frac{\mu}{\sqrt{(x-c)^2 + y^2}} - \frac{\mu'}{\sqrt{(x+c)^2 + y^2}}$$

Put $r' = q_1 + q_2$ and $r = q_1 - q_2$. T and V can be seen in the forms

$$T = \frac{1}{2}\left(q_1^2 - q_2^2\right)\left[\frac{\dot{q}_1^2}{q_1^2 - c^2} + \frac{\dot{q}_2^2}{c^2 - q_2^2}\right]$$

$$V = -\left[(\mu + \mu')\frac{q_1}{q_1^2 - q_2^2} + (\mu - \mu')\frac{q_2}{q_1^2 - q_2^2}\right]$$

So the problem is the Lionville's type.

Ex.3. If $T = \frac{1}{2}\left(q_1^2 + q_2^2\right)\left(\dot{q}_1^2 + \dot{q}_2^2\right)$ and $V = \frac{1}{q_1^2 + q_2^2}$, solve the problem completely using Liouville's approach.

Answer : One can easily identify

$$u_1(q_1) = q_1^2,\ u_2(q_2) = q_2^2,\ v_1(q_1) = 1,\ v_2(q_2) = 1,\ w_1(q_1) = 1,\ w_2(q_2) = 0$$

From (III.48)

$$\int \sqrt{\frac{v_1(q_1)}{d_1(q_1)}}\, dq_1 = \int \sqrt{\frac{v_2(q_2)}{d_2(q_2)}}\, dq_2 + \beta$$

where

$$d_1(q_1) = C_1 + hu_1(q_1) - w_1(q_1) = C_1 + hq_1^2 - 1$$

$$d_2(q_2) = C_2 + hu_2(q_2) - w_2(q_2) = -C_1 + h\,q_2^2 \quad (\because C_1 + C_2 = 0)$$

$$\therefore \quad \int \sqrt{\frac{1}{C_1 + hq_1^2 - 1}}\, dq_1 = \int \sqrt{\frac{1}{-C_1 + hq_2^2}}\, dq_2 + \beta$$

$$\text{or,} \quad \cos^{-1}\frac{q_1}{\sqrt{\frac{1 - C_1}{h}}} - \cos^{-1}\frac{q_2}{\sqrt{\frac{C_1}{h}}} = \text{constant} = \pi - C_0 \text{ (say)}$$

We thus arrive at the form

$$a^2 q_1^2 + b^2 q_2^2 + 2ab\, q_1 q_2 \cos C_0 = \sin^2 C_0$$

where $a = \sqrt{\frac{h}{1 - C_1}}$ and $b = \sqrt{\frac{h}{C_1}}$

The other equation can be similarly solved from (III.49).

Unit : IV ❑ Rotating Frames

IV.1 Basic equations : In this unit we shall discuss the motion of a particle relative to a rotating frame of reference. As mentioned earlier, a rotating frame is non-inertial in character and pseudo-forces generated out of it are to be carefully accounted for.

Consider two othogonal co-ordinate frames of reference (see Fig IV. 1 and Fig IV.2) namely S : OXYZ with unit axial vectors *i, j, k* and S′ : O′X′Y′Z′ of which S is attached to a rigid body R and S′ is held

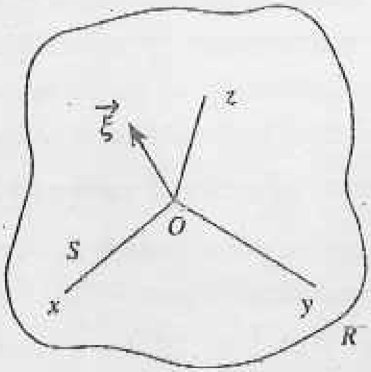


Fig IV.1 : Rotating coorrdinate system fixed in the rigid body R.

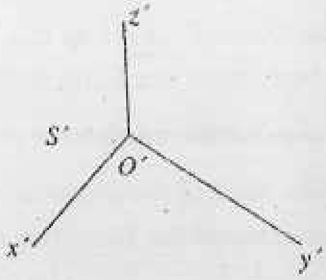


Fig IV.2 : Inertial Corrdinate system system fixed in space.

fixed in space. Now if the rigid body is rotating with and angular velocity $\vec{\omega}$ about a fixed axis through *O* then it is obvious that for a vector $\vec{\xi}$ fixed in *R*,. an observer positioned at *O* will see no change in the components (ξ_x, ξ_y, ξ_z) of $\vec{\xi}$ relative to *OX*, *OY*, *OZ*. However to an inertial observer at *O′*, the time-rate of change of $\vec{\xi}$ will appear as

$$\frac{d\vec{\xi}}{dt} = \left(\frac{d\xi_x}{dt}\vec{i} + \frac{d\xi_y}{dt}\vec{j} + \frac{d\xi_z}{dt}\vec{k}\right) + \xi_x\frac{d\vec{i}}{dt} + \xi_y\frac{d\vec{j}}{dt} + \xi_z\frac{d\vec{k}}{dt}$$

where $\frac{d\vec{i}}{dt} = \vec{w}\times\vec{i}$, $\frac{d\vec{j}}{dt} = \vec{w}\times\vec{j}$, $\frac{d\vec{k}}{dt} = \vec{w}\times\vec{k}$ are the induced velocities due to the angular velocity of the frame *S* relative to *S′*.

It therefore follows that

$$\left(\frac{d}{dt}\right)_{\text{fixed}} \vec{\xi} = \left(\frac{d}{dt}\right)_{\text{rot}} \vec{\xi} + \vec{\omega} \times \vec{\xi} \tag{IV.1}$$

where $\left(\frac{d}{dt}\right)_{\text{rot}}$ is to be identified with the time-rate as measured from the rotating frame S. Indeed Eq. (IV. 1) furnishes the typical motion of a rigid body being described by a combination of translation and rotation. The suffixed 'fixed' and 'rot' indicate the roles of a fixed observer in S and someone moving with the rigid body respectively.

In the following our aim would be to set up the governing equations of motion of a particle moving relative to a rotating frame. For concreteness, let us imagine the coordinate frame S' as set up in a fixed star thereby constituting an inertial frame and the rigid body to be our Earth itself. We neglect the orbital motion of Earth around the Sun and assume the Earth to be a perfect sphere. Now, the vector $\vec{\xi}$ can be typically assigned the roles of the distance vector $\vec{r}$ and velocity $\vec{v}$ of the particle as observed from the surface of the Earth (i.e. measued from the coordinate frame S'). Let P be the position of a particle at time t with $\vec{OP} = \vec{r}$ and $\vec{O'P} = \vec{r'}$. If $\vec{O'O} = \vec{a}$ then it is clear

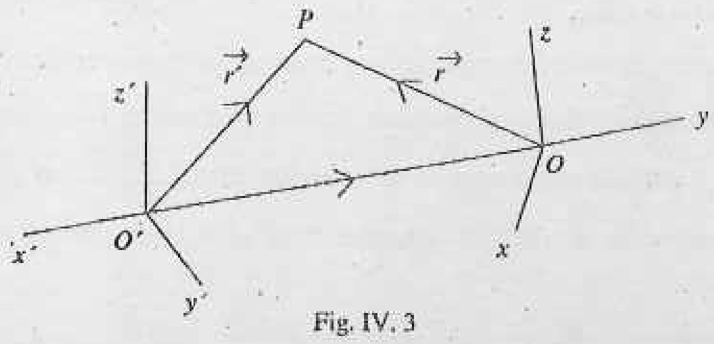


Fig. IV. 3

from Fig IV. 3 that $\vec{r'} = \vec{r} + \vec{a}$. As such the velocity of the particle at P as measured from O' would be

$$\vec{u} \equiv \left(\frac{d}{dt}\right)_{\text{fixed}} \vec{r'} = \left(\frac{d}{dt}\right)_{\text{fixed}} \vec{r} + \frac{d\vec{a}}{dt} \tag{IV.2}$$

To estimate $\left(\frac{d}{dt}\right)_{\text{fixed}}\vec{r}$ we use the relationship (IV. 1) between a fixed and a rotating frame. In consequence we get from (IV.2)

$$\vec{u} \equiv \left(\frac{d\vec{r}}{dt}\right)_{\text{rot}} + \vec{\omega}\times\vec{r} + \frac{d\vec{a}}{dt} \qquad \text{(IV.3)}$$

where we have replaced $\vec{\xi}$ by $\vec{r}$. The first term in the rhs of (IV.3) gives the velocity of the particle relative to the rotating frame S, the second term is the velocity due to the rotation of the *OXYZ* coordinate system and the third term is the so-called drag velocity. The drag velocity can be ignored if the distance vector between the points O and O' does not change with time.

To derive an expression for the acceleration of the particle at P as measured from O', we write from (IV. 3).

$$\left(\frac{d}{dt}\right)_{\text{fixed}}\vec{u} \equiv \left(\frac{d^2}{dt^2}\right)_{\text{fixed}}\vec{r} = \left(\frac{d}{dt}\right)_{\text{fixed}}\left(\frac{d\vec{r}}{dt}\right)_{\text{rot}} + \left(\frac{d}{dt}\right)_{\text{fixed}}\left(\vec{\omega}\times\vec{r}\right) + \frac{d^2\vec{a}}{dt^2}$$

...(IV.4)

Using (IV.1), with $\vec{\xi}$ replaced successively by the vectors $\left(\frac{d\vec{r}}{dt}\right)_{\text{rot}}$ and $\vec{\omega}\times\vec{r}$,

it follows that

$$\left(\frac{d}{dt}\right)_{\text{fixed}}\left(\frac{d\vec{r}}{dt}\right)_{\text{rot}} = \left(\frac{d^2\vec{r}}{dt^2}\right)_{\text{rot}} + \vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{\text{rot}}$$

$$\left(\frac{d}{dt}\right)_{\text{fixed}}\left(\vec{\omega}\times\vec{r}\right) = \left[\frac{d}{dt}\left(\vec{\omega}\times\vec{r}\right)\right]_{\text{rot}} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) \qquad \text{(IV.5)}$$

Substituting (IV.5) in (IV.4) we obtain

$$\left(\frac{d\vec{u}}{dt}\right)_{\text{fixed}} \equiv \left(\frac{d^2\vec{r}}{dt^2}\right)_{\text{fixed}} = \frac{d^2\vec{a}}{dt^2} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) + \left(\frac{d\vec{\omega}}{dt}\right)_{\text{rot}}\times\vec{r}$$

$$+2\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{\text{rot}} + \left(\frac{d^2\vec{r}}{dt^2}\right)_{\text{rot}} \qquad \text{(IV.6)}$$

Note that the first three terms in the right-hand-side of (IV.6) survive even when P is stationary relative to the rotating frame

$$S : \left(\frac{d\vec{r}}{dt}\right)_{rot} = \left(\frac{d^2\vec{r}}{dt^2}\right)_{rot} = 0$$

Ignoring the $\frac{d^2\vec{a}}{dt^2}$ term by assuming that the distance $O\vec{O'}$ is not changing with time and observing that in an inertial frame Newton's second law implies that $m\left(\frac{d\vec{u}}{dt}\right)_{fixed} = \vec{F}$, $\vec{F}$ representing the vector sum of the forces acting on the particle, we have from (IV.6).

$$m\left(\frac{d\vec{v}}{dt}\right)_{fixed} - \vec{F} = -2m\left(\vec{\omega}\times\vec{v}\right) - m\,\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) - m\,\dot{\vec{\omega}}\times\vec{r} \qquad \text{(IV.7)}$$

where we have set $\vec{v} = \left(\frac{d\vec{r}}{dt}\right)_{rot}$, $\frac{d\vec{v}}{dt} = \left(\frac{d^2\vec{r}}{dt^2}\right)_{rot}$ and $\dot{\vec{\omega}} = \left(\frac{d\vec{\omega}}{dt}\right)_{rot}$. The non-vanishing of the right-hand-side of (IV 7) is due to the presence of three types of forces namely, the coriolis force $2m\left(\vec{v}\times\vec{\omega}\right)$, the centrifugal force $m\,\vec{\omega}\times\left(\vec{r}\times\vec{\omega}\right)$ and the due to non-uniform rotation $-\,m\,\dot{\vec{\omega}}\times\vec{r}$. The latter can be neglected for a uniform rotation : $\frac{d\vec{\omega}}{dt} = 0$. Note that the form of the Coriolis term implies that it is always perpendicular to the direction of velocity and so it can never change the speed of a particle (except, of course, for its direction). It is also referred to as a deflecting force. Obviously the Coriolis force does not contribute to the energy equation.

Consider a typical three-dimensional motion with rotation about the z-axis (see Fig IV.4) :

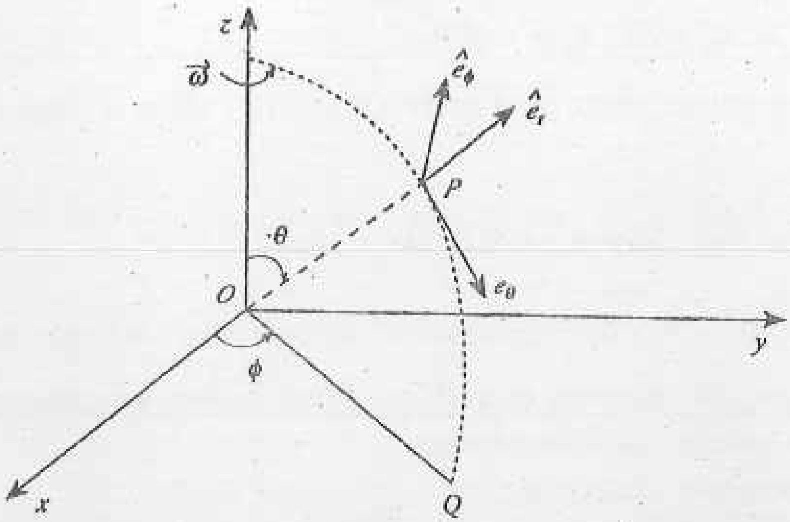


Fig. IV. 4

$$\vec{\omega} = \dot{\phi}\,\vec{k} = \dot{\phi}\left(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta\right)$$

$$\dot{\vec{\omega}} = \ddot{\phi}\,\vec{k} = \ddot{\phi}\left(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta\right)$$

where $\hat{e}_r, \hat{e}_\theta$ are the unit basis vectors in the $\vec{r}$ and $\vec{\theta}$ directions respectively. Note that the azimuthal plane OZQ is essentially two-dimensional.

In the rotating frame we can directly express

$$\vec{v} = \dot{r}\,\hat{e}_r + \left(r\dot{\theta}\right)\hat{e}_\theta$$

$$\dot{\vec{v}} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\hat{e}_\theta \qquad \text{(IV.8)}$$

$$\vec{\omega}\times\vec{r} = \phi\left(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta\right)\times r\hat{e}_r = r\sin\dot{\phi}\,\hat{e}_\phi$$

where $\hat{e}_\phi$ is the unit basis vector in the φ-direction. As a result, we have

$$\vec{\omega}\times\vec{v} = \dot{\phi}\left(\dot{r}\sin\theta - r\dot{\theta}\cos\theta\right)\hat{e}_\theta \qquad \text{(IV.9)}$$

$$\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) = -r\sin\theta\left(\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta\right)\dot{\theta}^2 \qquad \text{(IV.10)}$$

Further since $\vec{u} = \vec{v} \times \vec{\omega} \times \vec{r}$, we also have

$$u_r = \dot{r}, \quad u_\theta = r\dot{\theta}, \quad u_\phi = r \sin\theta\, \dot{\phi} \tag{IV.11}$$

Substituting (IV.8), (IV.9), (IV.10), (IV.11) in (IV.6) which in terms of $\vec{u}$ and $\vec{v}$ read

$$\frac{d\vec{u}}{dt} = \frac{d\vec{v}}{dt} + 2\vec{\omega} \times \vec{v} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) + \dot{\vec{\omega}} \times \vec{r} \tag{IV.12}$$

where $\frac{d^2\vec{a}}{dt^2}$ term has been ignored and suffixes 'fixed' and 'rot' have been dropped, we get the following expressions of the various components of the acceleration for a non-inertial rotating frame :

radial component of acceleration $f_r = \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\omega^2$

cross-radial component of acceleration $f_\theta = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) - r\omega^2 \sin\theta \cos\theta$

azimuthal component of acceleration $f_\phi = \frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2\omega \sin^2\theta\right)$ (IV.13)

IV.2 Some remarks on the Coriolis force : From (IV.7) the Coriolis force term is $-2m\vec{\omega} \times \vec{v}$ where $\vec{v} \equiv \dot{\vec{r}}$. Consider a flat rotating disc. For a particle moving (See Fiv IV. 5a)

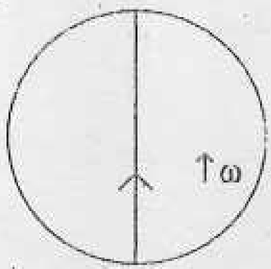


Fig. IV. 5a

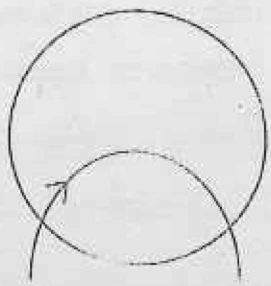


Fig. IV 5b

across a disc under no forces, an inertial observer (i.e. the one who is in a fixed frame) wil see it moving across a straight line according to Newton's law. However, in view of the fact that the disc is rotating, an observer stationed on the disc will view the particle taking a curved track due to the Coriolis force operating in a direction perpendicular to the motion of the particle (See Fig IV 5b). Note that the effect of the

Coriolis force is to bend the path of the particle to the right in the Northern Hemisphere and to the left in the Sourthern Hemisphere.

Assuming ω to be constant and neglecting the $\frac{d^2\vec{a}}{dt^2}$ term we can re-write Eq. (IV.6) as

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{\text{fixed}} = \left(\frac{d^2\vec{r}}{dt^2}\right)_{\text{rot}} + 2\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{\text{rot}} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) \qquad \text{(IV.14)}$$

For a particle moving under gravitation (i.e. the attractive force guided by Newton's law of gravitation) and also subjected to an additional force $\vec{F}$, the equation of motion is

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_{\text{fixed}} = m\vec{g} + \vec{F} \qquad \text{(IV.15)}$$

Thus from (IV.14) and (IV.15) we find

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_{\text{rot}} = m\vec{g} + \vec{F} - 2m\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{\text{rot}} - m\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) \qquad \text{(IV.16)}$$

In the laboratoiy when we measure the acceleration due to gravity what we determine is actually $\vec{g}^*$, the effective gravitational accleration given by

$$\vec{g}^* = \vec{g} - \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) \qquad \text{(IV. 17)}$$

The right-hand-side of (IV.17) is a combination of gravitational and centrifugal forces. Note that the horizontal and vertical components of $\vec{g}^*$ are $g^*_{\text{hor}} = \omega^2 r \sin\theta \cos\theta$ and $g^*_{\text{ver}} = g - \omega^2 r \sin^2\theta$.

At the pole $g^* = g$ while on the equator $g^* = g - \omega^2 r$.

Substituting (IV.17) in (IV.16) and writing $\vec{g}$ in place of $\vec{g}^*$ we get,

$$m\frac{d^2\vec{r}}{dt^2} = m\vec{g} + \vec{F} - 2m\vec{\omega}\times\frac{d\vec{r}}{dt} \qquad \text{(IV.18)}$$

where the suffix 'rot' has been dropped.

The components of $\vec{w}$ being $(0, \omega \sin \theta, \omega \cos \theta)$, where θ is the angle between the direction of earth's axis and $\vec{g}^*$ we find the Coriolis force to be given by

$$2m\,\vec{\omega} \times \frac{d\vec{r}}{dt} = 2m\omega(\dot{y}\cos\theta - \dot{z}\sin\theta, -\dot{x}\cos\theta, \dot{x}\sin\theta)$$

where $\frac{d\vec{r}}{dt} = (\dot{x}, \dot{y}, \dot{z})$ (IV.19)

IV.3 Foucault's pendulum : A useful device for observing the effects of the Coriolis force is the so called Foucault's pendulum. It is perfectly symmetric and designed to swing freely in any direction. Because of its symmetric nature, the periods of oscillation of a Foucault's pendulum in all directions are equal.

Neglecting the vertical component of the Coriolis force for it is very small compared to g, the equations of motion in the x and y directions are

$$\ddot{x} = -\frac{g}{l}x + 2w\dot{y}\cos\theta \qquad \text{(IV. 20)}$$

$$\ddot{y} = -\frac{g}{l}y - 2\omega\dot{x}\cos\theta$$

where we have used (IV.18) and (IV.19). One can see that the right-hand-sides of Eq. (IV.20) carry the contributions from the Coriolis acceleratoin in addition to the usual ones of the simple harmonic motion. In writing down (IV.20) we have also assumed that for small amplitude, the motion is nearly horizontal i.e. $\dot{z} \approx 0$.

A straighforward way to solve (IV.20) is to set $r = x + iy$ resulting in

$$\ddot{r} + 2ik\,\dot{r} + \omega_0^2 r = 0 \qquad \text{(IV.21)}$$

where $k = \omega\cos\theta$, $\omega_0^2 = g/l$. For $r = Ae^{\lambda t}$ where A and λ are constants we find $\lambda^2 + 2ik\lambda + \omega_0^2 = 0$ whose solutions are

$$\lambda = -ik \pm i\omega'$$

where $\omega'^2 = \omega_0^2 + k^2$. The solution of (IV.21) can thus be written as

$$r = C_1 e^{-i(k-\omega')t} + C_2 e^{-i(k+\omega')t} \qquad \text{(IV.22)}$$

An interesting particular case of (IV.22) corresponds to the choice of the integration constants $C_1 = C_2 = \frac{1}{2}a$:

$$r = a\,e^{-ikt}\cos\omega' t$$

$$\text{i.e.} \qquad x = a\cos kt\cos\omega' t,\ \ y = -a\sin kt\cos\omega' t \qquad \text{(IV.23)}$$

The interpretation of (IV.23) is that initially (i.e. at $t = 0$) the oscillation is in the x-direction. However, with passage of time, the amplitude of the y-coordinate grows at the expense of the x-coordinate that dampens. Overall the solution depicts an oscillation (of amplitude a) in a plane that is rotating with an angular velocity $-k$.

IV.4 The Lagrangian and velocity-dependent potential : If $\vec{u}$ be the velocity of a particle relative to the inertial frame C′ and $\vec{v} = \left(\frac{d\vec{r}}{dt}\right)_{\text{rot}}$ then from (IV.3) we have

$$\vec{u} = \vec{v} + \vec{\omega}\times\vec{r} \qquad \text{(IV.24)}$$

where the drag velocity has been omitted.

To set up a Lagrangian we need to ensure that the kinetic energy is measured relative to an inertial frame so that if $V(r)$ is the underlying potential then

$$L = \frac{1}{2}m\left|\vec{u}\right|^2 - V(r)$$

$$= \frac{1}{2}m\left[\vec{v} + \vec{\omega}\times\vec{r}\right]^2 - V(r)$$

$$= \frac{1}{2}m\left[\left|\vec{v}\right|^2 + \left(\vec{\omega}\times\vec{r}\right)^2 + 2\,\vec{v}.\left(\vec{\omega}\times\vec{r}\right)\right]^2 - V(r) \qquad \text{(IV.25)}$$

Exercise : Show that the equation of motion following from L is

$$m\frac{d\vec{v}}{dt} + \vec{\nabla}V = -2m\left(\vec{\omega}\times\vec{v}\right) - m\,\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) \qquad \text{(IV.26)}$$

where ω has been assumed constant.

Comparing (IV.26) with (IV.7) we find the two equations to be consistent for the potential force $\vec{F} = -\vec{\nabla} V$. We therefore conclude that an observer in a non-inertial rotating frame will feel the influence of a velocity-dependent potential U given by

$$U = V(r) - \frac{1}{2} m\left(\vec{\omega} \times \vec{r}\right)^2 - m\,\vec{v} \cdot \left(\vec{\omega} \times \vec{r}\right) \qquad \text{(IV.27)}$$

IV.5 Non-potential force : We now consider a situation in which the system is acted upon by non-potential forces apart from the potential forces :

$$Q'_j = Q'_j\,(q_k, \dot{q}_k, t) \qquad \text{(IV.28)}$$

where $j, k = 1, 2, \ldots n$. It is clear that the non-potential forces are to depend on generalized velocities for otherwise they can be identified with the potential forces defined by (III.8). Indeed in the presence of (IV.28). (III.8) is to be modified to

$$Q_j = -\frac{\partial V}{\partial q_j} + Q'_j \qquad \text{(IV.29)}$$

As such the Lagrange's equations of motion (III.17) take the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j} + Q'_j,\ j = 1, 2, \ldots\ldots n \qquad \text{(IV.30)}$$

We next look at the total energy $E = T + V$. T being $T = T(q_j, \dot{q}_j, t)$ we have

$$\frac{dT}{dt} = \sum_{j=1}^{n}\left(\frac{\partial T}{\partial q_j}\dot{q}_j + \frac{\partial T}{\partial \dot{q}_j}\ddot{q}_j\right) + \frac{\partial T}{\partial t}$$

$$= \frac{d}{dt}\sum_{j=1}^{n}\frac{\partial T}{\partial \dot{q}_j}\dot{q}_j + \sum_{j=1}^{n}\left[\frac{\partial T}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\right]\dot{q}_j + \frac{\partial T}{\partial t}$$

$$= \frac{d}{dt}\sum_{j=1}^{n}\frac{\partial T}{\partial \dot{q}_j}\dot{q}_j + \sum_{j=1}^{n}\left(\frac{\partial V}{\partial q_j} - Q'_j\right)\dot{q}_j + \frac{\partial T}{\partial t} \qquad \text{(IV.31)}$$

where we have used (IV.30)

Noting that the form of the kinetic energy T as given by (II.9) can be split up as

$$T = T_0 + T_1 + T_2 \qquad \text{(IV.32)}$$

where T_2, T_1 are quadratic and linear in generalized velocities respectively while T_0 is independent of it, we have by Euler's theorem of calculus

$$\sum_{j=1}^{n}\frac{\partial T_2}{\partial \dot{q}_j}\dot{q}_j = 2T_2.$$

$$\sum_{j=1}^{n} \frac{\partial T_1}{\partial \dot{q}_j} \dot{q}_j = T_1 \tag{IV.33}$$

As a result (IV.31) becomes

$$\frac{dT}{dt} = \frac{d}{dt}\left(2T_2 + T_1\right) + \sum_{j=1}^{n}\left(-\frac{\partial V}{\partial q_j} - Q_j'\right)\dot{q}_j + \frac{\partial T}{\partial t}$$

$$= 2\frac{dT}{dt} - \frac{d}{dt}\left(T_1 - 2T_0\right) + \frac{\partial T}{\partial t} + \frac{dV}{dt} - \frac{\partial V}{dt} - \sum_{j=1}^{n} Q_j' \dot{q}_j \tag{IV.34}$$

where we have used $\frac{dV}{dt} = \sum_{j=1}^{n} \frac{\partial V}{\partial q_j} \dot{q}_j + \frac{\partial V}{\partial t}$ and also $\frac{\partial T_0}{\partial \dot{q}_j} = 0$.

(IV.34) implies

$$\frac{dE}{dt} = \sum_{j=1}^{n} Q_j' \dot{q}_j + \frac{d}{dt}\left(T_1 + 2T_0\right) - \frac{\partial}{\partial t}(T - V) \tag{IV.35}$$

For a scleronomic system $T_1 = T_0 = 0$ and $\frac{\partial T}{\partial t} = 0$. If the potential energy too is not explicitly dependent upon time then $\frac{dE}{dt} = \sum_{j=1}^{n} Q_j' \dot{q}_j$. The latter is called the *power* of the non-potential forces.

Non-potential forces are called gyroscopic if the power is zero : $\sum_{j=1}^{n} Q'_j \dot{q}_j = 0$ and dissipative if the power is negative $\sum_{j=1}^{n} Q'_j \dot{q}_j < 0$. In the former case we have systems for which the nonpotential forces do not consume power while for the latter case we have systems experiencing dissipative forces which consume power. In dissipative systems, dissipative forces like friction are included, even through they sometimes do not do any work. The energy is generally lost through heat, sound etc.

On the other hand, for a scleronomic system, the Coriolis force is a gyroscopic force. Form the from (IV.19) it is clear that for the Coriolis force.

$$\vec{F}_j^{cor} = -2m_j\left(\vec{\omega} \times \frac{d\vec{r}_j}{dt}\right) = -2m(\vec{\omega} \times \vec{v}_j) \text{we have} \sum_{j=1}^{n} \vec{F}_j^{cor} . \vec{v}_j = 0.$$

We therefore conclude that the Coriolis forces of inertia are gyroscopic forces.

IV. 6 Examples :

1. Find the deflection of a freely falling body from the vertical caused by earth's rotation

It a particle is dropped from rest fum a height h to above the ground then the motion is described by

$$x = 0,\ y = 0,\ z = h - \frac{1}{z} g t^2$$

where the effect of the Coriolis force is neglected.

To find the effect of the Coriolis force to first order in ω, we substitute for $\frac{d\vec{r}}{dt}$ in (IV.18) the zero-order expressions namely

$$\dot{x} - 0,\quad \dot{y} = 0,\quad \dot{z} = -gt$$

Then using (IV.19), (IV.18) reads componentwise

$$m\ddot{x} = 2m\omega gt \sin\theta,\ m\ddot{y} = 0,\ m\ddot{z} = -gt$$

We thus find that the particle will hit the ground $z = 0$ (i.e. $t = \sqrt{\frac{2h}{g}}$ from $z = h - \frac{1}{2} g t^2$) in the eastern direction at a distance of

$$x = \omega g \frac{1}{3} \sin\theta \left(\frac{2h}{g}\right)^{3/2}$$

i.e. $$x = \frac{1}{3}\omega \left(\frac{8h^3}{g}\right)^{1/2} \sin\theta$$

where we have integrated the above expression for $\ddot{x}$ and used $x = 0,\ \dot{x} = 0$ at $t = 0$. We conclude that there will be an easterly deviation from the vertical of amount $\frac{1}{3}\omega \sin\theta \sqrt{\frac{8h^3}{g}}$.

2. A bead of mass m slides freely along a smooth circular wire which is rotating with an angular velocity $\vec{\omega}$ about its fixed vertical diameter. Derive and discuss the energy conservation equation. Take a to be the radius of the wire and set $\omega = \sqrt{\frac{ng}{a}}$ where n is a parameter.

As shown in Fig IV.6, we have

$$\vec{\omega} = \omega\,\hat{e}_z$$

$$= \omega\left(-\cos\theta\,\hat{e}_r + \sin\theta\,\hat{e}_\theta\right)$$

The centrifugal force has the magnitude

$$= m\omega^2\left|\overrightarrow{NP}\right|$$

$$= m\omega^2 a\sin\theta$$

and acts horizontally outwards along $\overrightarrow{NP}$.

Obviously this force has a relevance in rotating frames only.

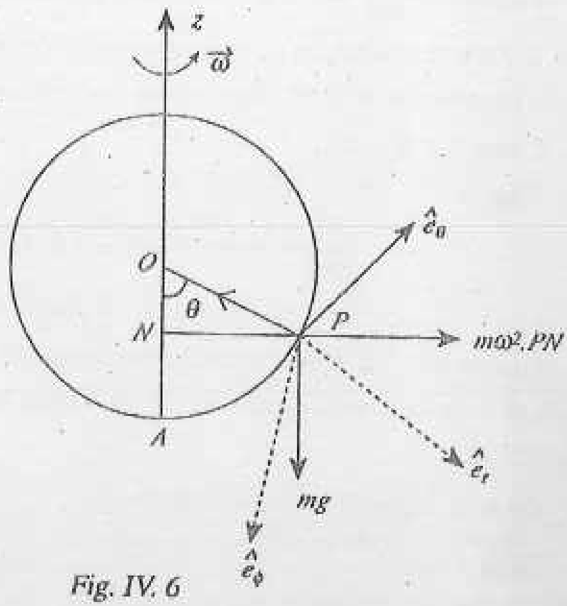


Fig. IV. 6

From the second relation of (IV.13), the cross-radial equation of motion reads

$$m\cdot\frac{1}{a}\frac{d}{dt}\left(a^2\dot{\theta}\right) = -mg\sin\theta + (m\omega^2 a\sin\theta)\cos\theta$$

$$\text{or,}\quad a\ddot{\theta} = -g\sin\theta + ng\sin\theta\cos\theta$$

where we have put $a\omega^2 = ng$. Integration gives

$$\frac{1}{2}ma^2\dot{\theta}^2 + \frac{mga}{2n}(1-n\cos\theta)^2 = \text{constant}$$

This is essentially the energy conservation equation in a rotating frame with $\frac{1}{2}ma^2\dot{\theta}^2$ representing the kinetic energy and the potential energy is given by

$$V(n,\theta) = \frac{mga}{2n}(1-n\cos\theta)^2$$

We also get

$$\frac{dV}{d\theta} = mg\,a\sin\theta(1-n\cos\theta)$$

$$\frac{d^2V}{d\theta^2} = mg\,a[\cos\theta(1-n\cos\theta) + n\sin^2\theta]$$

Therefore $\frac{dV}{d\theta} = 0$ at $\theta = 0$ (namely, the lowest point A) and also at $\cos\theta = \frac{1}{n}$ for $n > 1$. Now a minimum of V corresponds to the position of relative stable equilibrium and a maximum of V corresponds to the position of relative unstable equilibrium.

Case 1 : $\theta = 0$

We get

$$\frac{d^2V}{d\theta^2} = mga(1-n) > 0 \text{ if } n < 1$$

$$\Rightarrow \theta = 0 \text{ for } n < 1 \text{ is a minimum for } V.$$

Hence $\theta = 0$ is a position of relative stable equilibrium if $n < 1$. If however $n > 1$, then $\theta = 0$ is a maximum for V and so $\theta = 0$ is a position of relative unstable equilibrium.

Case 2 : $\cos\theta = \frac{1}{n}$ $(n > 1)$

Here

$$\frac{d^2V}{d\theta^2} = n\left(1 - \frac{1}{n^2}\right) > 0$$

Therefore $\cos\theta = \frac{1}{n}$ is a position of relative stable equilibrium. We conclude that new stable solutions are created at $\cos\theta = \frac{1}{n}$ as n exceeds beyond the critical value of $n = 1$. Such a phenomenm is called bifurcation in the language of differential equations.

IV.7 : Noholonomic Constraints : Nonholonomic constraints are characterized by inequalities or non-integrable equations. Most velocity dependent forces are nonholonomic.

Consider the rolling of a sphere on a plane. Let $\vec{V}$ be the translational velocity and $\vec{\omega}$ be the angular velocity of rotation. The velocity of the point of contact may be obtained from

$$\vec{v} = \vec{V} + \vec{\omega} \times \vec{r}$$

Putting $\vec{r} = -a\hat{n}$, a is the radius of the sphere and $\hat{n}$ is the unit vector along the normal to the sphere. The condition for no sliding at the point of contract is

$$\vec{V} - a\,\vec{\omega} \times \vec{n} = 0$$

which connot be integrated. The reason is that $\vec{\omega}$ is not generally expresible as the time derivative of a coordinate. The above constraint is nonholonomic.

As another example consider the motion of a coin which is vertical and rolling on the xy-plane. Its orientation is given by the two angles θ and ϕ (see figure IV.7). These correspond to the two degrees of freedom of the coin whose radius is say r.

Let the coordinate of the centre of the coin, namely C, when projected on the xy-plane be (x, y) which is the point of contact of the coin with the xy-plane. The relevant equations are

$$v = r\dot{\phi}$$

$$\dot{x} = -v \cos\theta$$

$$\dot{y} = -v \sin\theta$$

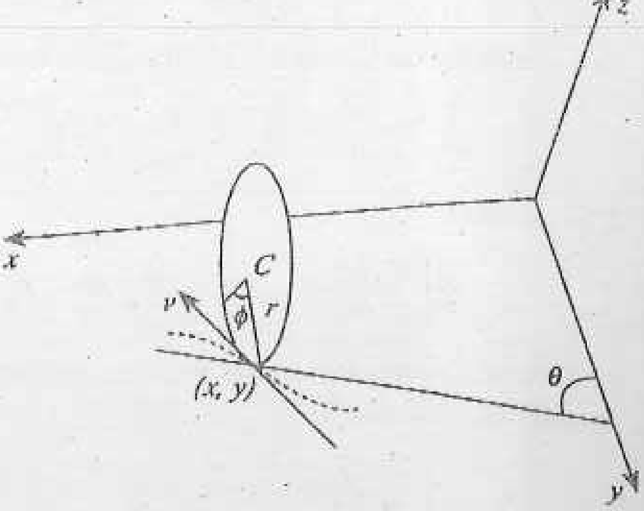


Fig. IV.7

In differential terms these are

$$dx = -r\cos\theta \, d\phi$$

$$dy = -r\sin\theta \, d\phi$$

That these are non-integrable follows from the fact that for integrability we would have

$$dx + r\cos\theta \, d\phi = df(x, \theta, \phi) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \phi} d\phi$$

In particular $\frac{\partial f}{\partial \theta} = 0$, $\frac{\partial f}{\partial \phi} = r\cos\theta$ implying $\frac{\partial^2 f}{\partial\theta\,\partial\phi} \neq \frac{\partial^2 f}{\partial\phi\,\partial\theta}$

which is not true for any smooth function. So f does not exist.

Similarly for the other differential equation. We conclude that the above problem is nonholonomic in nature.

Unit : V ❑ Hamiltonian and Poisson Bracket

V.1 : The Hamiltonian : Consider the total time rate of change in the Lagrangian $L(q_i, \dot{q}_i, t)$, $i = 1, 2, \ldots n$:

$$\frac{dL}{dt} = \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \dot{q}_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} + \frac{\partial L}{\partial t}$$

Since the second term in the rhs can be written as

$$\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} = \frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i\right) - \frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i$$

we have

$$\frac{d}{dt}\left[\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L\right] = -\sum_{i=1}^{n}\left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right]\dot{q}_i - \frac{\partial L}{\partial t}$$

If the coordinates q_i's obey Lagrangian equations of motion then the above equation boils down to

$$\frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L\right) = -\frac{\partial L}{\partial t} \qquad \text{(V.1)}$$

We now define the generalized momentum p_i associated with the generalized co-ordinate q_i to be

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1, 2, \ldots n \qquad \text{(V.2)}$$

Note that since the kinetic energy in terms of the contesian velocities is $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$, the components of the linear momentum in terms of T are simply $\frac{\partial T}{\partial \dot{x}}$, $\frac{\partial T}{\partial \dot{y}}$ and $\frac{\partial T}{\partial \dot{z}}$. It is therefore appropriate to view $\frac{\partial T}{\partial \dot{q}_i}$ as a kind of generalized memontum which when the potential is a function of position only represents $\frac{\partial L}{\partial \dot{q}_i}$ as in (V.2). Thus (V.1) reads

$$\frac{d}{dt}\left(\sum_{i=1}^{n} p_i \dot{q}_i - L\right) = -\frac{\partial L}{\partial t} \qquad \text{(V.3)}$$

The construction of the Hamiltonian from a given Lagrangian relies upon a Legendre transformation from $\dot{q}_i$ to p_i. In other words we introduce a function H defined in the manner

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L \tag{V.4}$$

so that from (V.3)

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \tag{V.5}$$

The function H is known as Hamilton's function or more precisely the Hamiltonian. It is at once implied from (V. 5) that if t does not appear explicity in L then H is reduced to a constant in time.

We have aheady noted in Unit (III) that for a conservative, scleronomic system the quantity $\sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$ is a constant and that if the kinetic energy is a homogeneous quadratic function of the velocities, the total energy is constant as well. In mathematical language this means $L = T - V$ and

$$\begin{aligned}\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i = \sum_{i=1}^{n} p_i \dot{q}_i \\ &= \sum_{i=1}^{n} \frac{\partial (T-V)}{\partial \dot{q}_i} \dot{q}_i \\ &= \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} \dot{q}_i \\ &= 2T\end{aligned}$$

where we have employed Euler's theorem in the last step. As such

$$\begin{aligned}H &= 2T - (T - V) \\ &= T + V \\ &= \text{total energy}\end{aligned}$$

H can also be interpreted as "generalized energy".

V.2 : Hamilton's Canonical Equations : Just as we wrote down Lagrange's equations of motion for a dynamical system with n degrees of freedom, similarly we can generate Hamilton's canonical equations of motion from the definition of the Hamiltonian. First of all, taking differential of (V.4) we get

$$dH = \sum_{i=1}^{n}\left[\dot{q}_i\, dp_i + \left(p_i - \frac{\partial L}{\partial \dot{q}_i}\right)d\dot{q}_i - \frac{\partial L}{\partial q_i}\, dq_i\right] - \frac{\partial L}{\partial t}\, dt$$

$$= \sum_{i=1}^{n}\left(\dot{q}_i\, dp_i - \frac{\partial L}{\partial q_i}\, dq_i\right) - \frac{\partial L}{\partial t}\, dt \qquad \text{(V.6)}$$

where we have exploited (V.2).

Next, looking upon H as a function of q_i, p_i and t i.e. $H = H(q_i, p_i, t)$, $i = 1, 2, \ldots, n$, dH produces

$$dH = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i}\, dq_i + \frac{\partial H}{\partial p_i}\, dp_i\right) + \frac{\partial H}{\partial t}\, dt \qquad \text{(V.7)}$$

Comparing (V.6) and (V.7) yields

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = \frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial q_i}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \qquad \text{(V.8)}$$

The equations of the $2n$ variables (q_i, p_i) are thus subject to

$$\dot{q}_i \equiv \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i \equiv \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad \text{(V.9)}$$

Equations (V.9) are called Hamilton's canonical equations of motion. The name 'canonical equations' anises from the fact that the investigation of motions subjected to the influence of a potential is being reduced to the examination of differential equations of the form (V.9). Interestingly the set of equations (V.9) is invariant under the replacements $q_i \to p_i$ and $p_i \to -q_i$.

Notice that using (V.9) $\frac{dH}{dt}$ becomes

$$\frac{dH}{dt} = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i\right) + \frac{\partial H}{\partial t}$$

$$= \frac{\partial H}{\partial t}$$

Implying that if H does not depend upon t explicitly then it is a sconstant of motion.

Finally we note that in Lagrangian formalism we are required to solve n second-order differential equations for a system with n degrees of freedom. However, in Hamiltonian dynamics we have at hand $2n$ first-order differential equations where the unknown functions are q_i and p_i as functions of the time t. The almost symmetrical appearance of q and p, as already noted above, facilitates development of formal theories such as conical transformations, action-angle variables etc. to which we shall come later. We refer to (q_i, p_i) as the set of canical variables. The cooridnates $(q_1, q_2, \ldots q_n)$ and the momenta $(p_1, p_2, \ldots p_n)$ constitute) a $2n$-dimensional space called the phase space. The Hamilton's equations describe the evolution of such a phase space. The Lagrangian $L(q_1, q_2, \ldots q_n ; \dot{q}_1, \dot{q}_2, \ldots \dot{q}_n ; t)$, on the other hand is interprected in the configuration space with q_i and $\dot{q}_i$ denoting respectively the coordinates and velocities at a specific point at time t.

Remark : For the kinetic energy function homogeneous in the second degree of the generalized velocities as given by (II.11), the Lagrangian reads

$$L = \frac{1}{2}\sum_{j,k=1}^{n} a_{ik}\,\dot{q}_i\,\dot{q}_k - V(q)$$

where $q = (q_1, q_2, \ldots q_n)$ and $a_{ik} = a_{ki}$. Lagrange's equations are

$$\frac{d}{dt}\left[\sum_{j=1}^{n} a_{kj}\,\dot{q}_j\right] = \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial a_{ij}}{\partial q_k}\dot{q}_i\,\dot{q}_j - \frac{\partial V(q)}{\partial q_k}$$

Inverting the equations $p_j = \frac{\partial L}{\partial \dot{q}_j} = \sum_{i=1}^{n} a_{ji}\,(q)\dot{q}_j$ we have

$$\dot{q}_i = a_{ij}^{-1}\,p_j \qquad \left(\text{in general } a_{ij}^{-1} \neq \left(a_{ij}\right)^{-1}\right)$$

where a_{ij}^{-1} are the elements of the matrix A^{-1} :

$$L = \frac{1}{2}\,\dot{q}^T A\, q - V(q),\ T \rightarrow \text{transpose}$$

and we have suppressed the summation sign. Lagrange's equation can thus be expressed as

$$\dot{p}_k = \frac{1}{2}\left(a_{is}^{-1}\, p_s\right)\frac{\partial a_{ij}}{\partial q_k}\left(a_{jm}^{-1}\, p_m\right) - \frac{\partial V}{\partial q_k}$$

$$= \frac{1}{2}\, p^T A^{-1}\,\frac{\partial A}{\partial q_k}\, A^{-1} p - \frac{\partial V}{\partial q_k}$$

$$= -\frac{1}{2}\, p^T\,\frac{\partial A^{-1}}{\partial q_k}\, p - \frac{\partial V}{\partial q_k}$$

Thus the Hamiltonian can be defined as

$$H(q, p) = \frac{1}{2}\, p^T A^{-1} p + V$$

with $$\dot{p}_k = -\frac{\partial H}{\partial q_k},\ \dot{q}_k = \frac{\partial H}{\partial p_k}$$

Ex. 1. Solve the plane pendulum problem using the Hamiltonian approach.

Here the Lagrangian is (see unit III)

$$L = \frac{1}{2}\, ml^2\dot{\theta}^2 + mgl\cos\theta$$

$$\therefore\quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

So the Hamiltonian is

$$H = p_\theta\,\dot{\theta} - L = \frac{1}{2}\, ml^2\,\dot{\theta}^2 - mgl\cos\theta$$

But H needs to be defined in terms of proper variables θ and p_θ. Consequently we re-write H as

$$H = \frac{p_\theta^{\,2}}{2ml^2} - mgl\cos\theta$$

using $p_\theta = ml^2\,\dot{\theta}$. Hamilton's equation are

$$\frac{d\theta}{dt} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}$$

$$\frac{dp_\theta}{dt} = -\frac{\partial H}{\partial \theta} = -mgl\sin\theta$$

We see that t does not appear explicitly in H. So H is a constant of motion :

$$E = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta$$

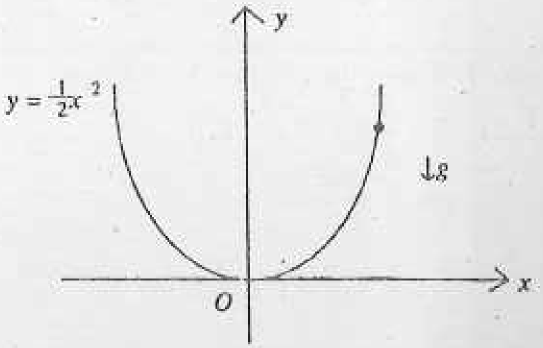
where the energy E is the constant value of the Hamiltonian.

Ex. 2. Examine the motion of a particle sliding on a parabolic wire.

Consider the sliding of a particle on a wire bent to a form of a parabola. The particle is acted upon by gravity only. We ignore friction.

Let x be the generalized coordinate and the parabola be given by the from $y = \frac{1}{2}x^2$.

The Lagragian is



$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

$$= \frac{1}{2}m\left(1 + x^2\right)\dot{x}^2 - \frac{mg}{2}x^2$$

$$\therefore\ p = \frac{\partial L}{\partial \dot{x}} = m\left(1 + x^2\right)\dot{x}$$

So $H = T + V = \dfrac{p^2}{2m\left(1 + x^2\right)} + \dfrac{mg}{2}x^2$

It implies

$$\dot{x} = \frac{p}{m\left(1 + x^2\right)}, \quad \dot{p} = -x\left[\frac{p^2}{m\left(1 + x^2\right)} + mg\right]$$

Noe that $p \neq m\dot{x}$.

Ex. 3. The Lagrangian for a free particle in terms of paraboloidal coordinates (ξ, η, ϕ) is

$$L = \frac{1}{2}m\left(\xi^2 + \eta^2\right)\left(\dot{\xi}^2 + \dot{\eta}^2\right) + \frac{1}{2}m\xi^2\eta^2\dot{\phi}^2$$

Set up the Hamiltonian.

The paraboloidal coordinates havre already been defined in Unit I.

Here

$$p_\xi = \frac{\partial L}{\partial \dot{\xi}} = m\left(\xi^2 + \eta^2\right)\dot{\xi}$$

$$p_\eta = \frac{\partial L}{\partial \dot{\eta}} = m\left(\xi^2 + \eta^2\right)\dot{\eta}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\xi^2\eta^2\dot{\phi}$$

Hence

$$H = p_\xi\,\dot{\xi} + p_\eta\,\dot{\eta} + p_\phi\,\dot{\phi} - L$$

$$= \frac{1}{2m}\left[\frac{p_\xi^2 + p_\eta^2}{\xi^2 + \eta^2} + \frac{p_\phi^2}{\xi^2\eta^2}\right]$$

Ex. 4. Solve the planetary problem.

Here $T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$ [we put $m = 1$]

$$V = -\frac{\mu}{r}$$

$$\therefore\ L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{\mu}{r}$$

As such

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

Hence the Hamiltonian is

$$H = T + V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{\mu}{r}$$

$$= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{\mu}{r}$$

So we deduce

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{1}{m}p_r,\ \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{\mu}{r^2}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2},\ \ \dot{p}_\theta = 0$$

Now $\dot{p}_\theta = 0$ implies

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$

or, $\quad mr^2\dot{\theta} = \text{constant} = l \text{ (say)}$

Eliminating p_r, p_θ and θ from the above equations we arrive at the form

$$m\ddot{r} = \frac{l^2}{mr^3} - \frac{\mu}{r^2}$$

the consequence of which have been discussed in Unit III in connection with Routh's procedure of ignoration of coordinates.

Ex. 5. If all the coordinates of a system are cyclic prove that the coordinates may be found out by integration. Prove further that if the system be scleronomous then the coordinates are linear functions of time.

Since the coordinates are cyclic

$$H = H(p_i, t), \qquad i = 1, 2, \ldots, n$$

$$\therefore \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} = 0, \quad i = 1, 2, \ldots, n$$

So

$$p_i = \text{constant} = \beta_i \text{ (say)}$$

Further

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \phi_i(p_1, p_2, \ldots p_n; t) \text{ (say)}$$

$$= \phi_i(\beta_1, \beta_2, \ldots \beta_n;\ t)$$

$$\therefore \quad q_i = \int \phi_i(\beta_1, \beta_2, \ldots \beta_n;\ t)dt + \alpha_i$$

where α_i's are constants of integration. Thus the coordinates may be found out *by* integration.

For a scleronomous system t does not appear in ϕ_i and so

$$q_i = \phi_i(\beta_1, \beta_2, \ldots \beta_n)t + \alpha_i, \quad i = 1, 2, \ldots, n$$

which are linear in time.

Ex. 6. Write down Hamilton's equations in spherical polar coordinates for

$$L = \frac{1}{2} m \vec{r}^{\,2} + \frac{k}{r},\ k \text{ a constant.}$$

Here $A = m\,1$ and $A^{-1} = m^{-1}\,1$ Consequently

$$H = \frac{1}{2} p^T A^{-1} p + V = \frac{1}{2m} \vec{p}^{\,2} - \frac{k}{r}$$

In spherical polar coordinates

$$L = \frac{1}{2} m\left(\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2 \sin^2\theta\,\dot{\phi}^2\right) + \frac{k}{r}$$

So the matrices A and A^{-1} and given by

$$A = \begin{pmatrix} m & 0 & 0 \\ 0 & mr^2 & 0 \\ 0 & 0 & mr^2 \sin^2\theta \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{mr^2} & 0 \\ 0 & 0 & \frac{1}{mr^2 \sin^2\theta} \end{pmatrix}$$

The Haniltonian reads

$$H = \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2} p_\theta^2 + \frac{1}{r^2 \sin^2\theta} p_\phi^2\right) - \frac{k}{r}$$

Hamilton's equations $\dot{p}_r = -\frac{\partial H}{\partial r},\ \dot{p}_\theta = -\frac{\partial H}{\partial \theta},\ \dot{p}_\phi = -\frac{\partial H}{\partial \phi}$

translate to

$$m\ddot{r} = \frac{\Lambda^2}{mr^3} - \frac{k}{q^2},\ \frac{d}{dt}\left(mr^2\dot{\theta}\right) = \frac{\cos\theta}{mr^2 \sin^3\theta} p_\phi^2,\ \dot{p}_\phi = 0$$

where $\Lambda^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}$ and $p_r = m\dot{r},\ p_\theta = mr^2\dot{\theta},\ p_\phi = mr^2 \sin^2\theta\,\dot{\phi}$ from $\begin{pmatrix} p_r \\ p_\theta \\ p_\phi \end{pmatrix} = A \begin{pmatrix} \dot{r} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix}$.

Problems :

(1) If the Hamiltanian of a dynamical system is given by $H = p_1q_1 - p_2q_2 - aq_1^2 + bq_2^2$ where a, b are constants, show that $\frac{p_2 - bq_2}{q_1}$ = constant.

(2) If $H = qp^2 - pq + bp$. Find q and p as functions of t. Here b is a constant.

(3) The Lagrangian for the motion of a particle in a rotating frame is

$$L = \frac{1}{2} m \vec{r}^{\,2} + \frac{1}{2} m\omega^2\left(x_1^2 + x_2^2\right) + m\omega\left(x_1\dot{x}_2 - x_2\dot{x}_1\right)$$

Find the Hamiltonian.

V. 3 Poisson bracket : Consider some function $f(q, p, t)$ of the canonical variables q_i and p_i. Then for $i = 1, 2, \ldots n$

$$\frac{df}{dt} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial f}{\partial p_i}\frac{dp_i}{dt}\right) + \frac{\partial f}{\partial t}$$

$$= \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right) + \frac{\partial f}{\partial t} \qquad \text{(V.10)}$$

We next introdure a quantity $\{u, v\}$ inolving the functions $u = u(q_i, p_i, t)$ and $v = v(q_i, p_i, t)$ and defined by

$$\{u, v\} = \sum_{i=1}^{n}\left(\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial q_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right) \qquad \text{(V. 11)}$$

Then (V.10) acquires the from

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t} \qquad \text{(V.12)}$$

It states that if f is a constant of motion then $\frac{\partial f}{\partial t} + \{f, H\} = 0$. Furthermore, if f does not involve t explicitly then $\{f, H\} = 0$

We also have

$$\{q_i, H\} = \sum_{j=1}^{n}\left(\frac{\partial q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial q_i}{\partial p_j}\frac{\partial H}{\partial q_j}\right)$$

$$= \sum_{j=1}^{n}\delta_{ij}\frac{\partial H}{\partial p_j} = \frac{\partial H}{\partial p_i} = \dot{q}_i \qquad \text{(V.13a)}$$

Similarly

$$\{p_i, H\} = \dot{p}_i \tag{V.13b}$$

So Hamilton's equations assume the form

$$\frac{dq_i}{dt} = \{q_i, H\}$$
$$\frac{dp_i}{dt} = \{p_i, H\} \tag{V.14}$$

where $i = 1, 2, \ldots, n$. The quantity $\{u, v\}$ defined by (V.11) is called the Poisson bracket of two dynamical variables $u(q_i, p_i, t)$ and $v(q_i, p_i, t)$ and plays an important role in Hamiltonian mechanics. A trivial consequence of (V.11) is the result

$$\{q_i, p_j\} = \delta_{ij}$$

V.4 Properties of Poison bracket : Poisson bracket satisfies several interesting properties. We give below a few of these with proofs.

1., Linearity : $\{u_1 + u_2, v\} = \{u_1, v\} + \{u_2, v\}$

$\{Cu, v\} = C\{u, v\}$, C a constant

2. Antisymmetry : $\{u, v\} = -\{v, u\}$

3. Product rule : $\{u, vw\} = \{u, v\}w + v\{u, w\}$

4. Jacobi identity : $\{u, \{v, w\}\} + \{v, \{w, u\}\} + \{w, \{u, v\}\} = 0$

Proofs :

$$\text{(1) } \{u_1 + u_2, v\} = \sum_{i=1}^{n}\left[\frac{\partial(u_1+u_2)}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial(u_1+u_2)}{\partial p_i}\frac{\partial v}{\partial q_i}\right]$$

$$= \sum_{i=1}^{n}\left[\left(\frac{\partial u_1}{\partial q_i} + \frac{\partial u_2}{\partial q_i}\right)\frac{\partial v}{\partial p_i} - \left(\frac{\partial u_1}{\partial p_i} + \frac{\partial u_2}{\partial p_i}\right)\frac{\partial v}{\partial q_i}\right]$$

$$= \sum_{i=1}^{n}\left[\left(\frac{\partial u_1}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u_1}{\partial p_i}\frac{\partial v}{\partial q_i}\right)\right] + \sum_{i=1}^{n}[1 \to 2]$$

$$= \{u_1, v\} + \{u_2, v\}$$

$$\{Cu, v\} = \sum_{i=1}^{n}\left[\frac{\partial(Cu)}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial(Cu)}{\partial p_i}\frac{\partial v}{\partial q_i}\right]$$

$$= \sum_{i=1}^{n}\left[C\frac{\partial(u)}{\partial q_i}\frac{\partial v}{\partial p_i} - C\frac{\partial(u)}{\partial p_i}\frac{\partial v}{\partial q_i}\right]$$

$$= C\{u, v\}$$

$$(2)\quad \{u, v\} = \sum_{i=1}^{n}\left(\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right)$$

$$= -\sum_{i=1}^{n}\left(\frac{\partial v}{\partial q_i}\frac{\partial u}{\partial p_i} - \frac{\partial v}{\partial p_i}\frac{\partial u}{\partial q_i}\right)$$

$$= -\{v, u\}$$

$$(3)\quad \{u, vw\} = \sum_{i=1}^{n}\left[\frac{\partial u}{\partial q_i}\frac{\partial (vw)}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial (vw)}{\partial q_i}\right]$$

$$= \sum_{i=1}^{n}\left[\frac{\partial u}{\partial q_i}\left(\frac{\partial v}{\partial p_i}w + v\frac{\partial w}{\partial p_i}\right) - \frac{\partial u}{\partial p_i}\left(\frac{\partial v}{\partial q_i}w + u\frac{\partial w}{\partial q_i}\right)\right]$$

$$= \sum_{i=1}^{n}\left(\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right)w + v\sum_{i=1}^{n}\left(\frac{\partial u}{\partial q_i}\frac{\partial w}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial w}{\partial q_i}\right)$$

$$= \{u, v\}w + v\{u, w\}$$

$$(4)\quad \{u, \{v, w\}\} + \{v, \{w, u\}\}$$

$$= \{u, \{v, w\}\} - \{v, \{u, w\}\}$$

$$= \sum_{i=1}^{n}\left[\left\{u, \frac{\partial v}{\partial q_i}\frac{\partial w}{\partial p_i} - \frac{\partial v}{\partial p_i}\frac{\partial w}{\partial q_i}\right\} - \left\{v, \frac{\partial u}{\partial q_i}\frac{\partial w}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial w}{\partial q_i}\right\}\right]$$

$$= \sum_{i=1}^{n}\left[\left\{u, \frac{\partial v}{\partial q_i}\frac{\partial w}{\partial p_i}\right\} - \left\{u, \frac{\partial w}{\partial q_i}\frac{\partial v}{\partial p_i}\right\} - \left\{v, \frac{\partial u}{\partial q_i}\frac{\partial w}{\partial p_i}\right\} + \left\{v, \frac{\partial w}{\partial q_i}\frac{\partial u}{\partial p_i}\right\}\right]$$

$$= \sum_{i=1}^{n}\left[\left\{u, \frac{\partial v}{\partial q_i}\right\}\frac{\partial w}{\partial p_i} + \frac{\partial v}{\partial q_i}\left\{u, \frac{\partial w}{\partial p_i}\right\} - \left\{u, \frac{\partial w}{\partial q_i}\right\}\frac{\partial v}{\partial p_i} - \frac{\partial w}{\partial q_i}\left\{u, \frac{\partial v}{\partial p_i}\right\}\right.$$

$$\left. - \left\{v, \frac{\partial u}{\partial q_i}\right\}\frac{\partial w}{\partial p_i} - \frac{\partial u}{\partial q_i}\left\{v, \frac{\partial w}{\partial p_i}\right\} + \left\{v, \frac{\partial w}{\partial q_i}\right\}\frac{\partial u}{\partial p_i} + \frac{\partial w}{\partial q_i}\left\{v, \frac{\partial u}{\partial p_i}\right\}\right]$$

$= X + Y$ (say)

where we have used the product rule and defined X, Y as

$$X = \sum_{i=1}^{n}\left[\left\{u, \frac{\partial v}{\partial q_i}\right\}\frac{\partial w}{\partial p_i} - \frac{\partial w}{\partial q_i}\left\{u, \frac{\partial v}{\partial p_i}\right\} - \left\{v, \frac{\partial u}{\partial q_i}\right\}\frac{\partial w}{\partial p_i} + \frac{\partial w}{\partial q_i}\left\{v, \frac{\partial u}{\partial p_i}\right\}\right]$$

$$Y = \sum_{i=1}^{n}\left[\frac{\partial v}{\partial q_i}\left\{u, \frac{\partial w}{\partial p_i}\right\} - \left\{u, \frac{\partial w}{\partial q_i}\right\}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial q_i}\left\{v, \frac{\partial w}{\partial p_i}\right\} + \left\{v, \frac{\partial w}{\partial q_i}\right\}\frac{\partial u}{\partial p_i}\right]$$

Now

$$X = \sum_{i=1}^{n}\left[\left(\left\{u, \frac{\partial v}{\partial q_i}\right\} - \left\{v, \frac{\partial u}{\partial q_i}\right\}\right)\frac{\partial w}{\partial p_i} - \frac{\partial w}{\partial q_i}\left(\left\{u, \frac{\partial v}{\partial p_i}\right\} - \left\{v, \frac{\partial u}{\partial p_i}\right\}\right)\right]$$

$$= \sum_{i=1}^{n}\left[\left(\left\{u, \frac{\partial v}{\partial q_i}\right\} + \left\{\frac{\partial u}{\partial q_i}, v\right\}\right)\frac{\partial w}{\partial p_i} - \frac{\partial w}{\partial q_i}\left(\left\{u, \frac{\partial v}{\partial p_i}\right\} + \left\{\frac{\partial u}{\partial p_i}, v\right\}\right)\right]$$

$$= \sum_{i=1}^{n}\left[\frac{\partial}{\partial q_i}\{u, v\}\frac{\partial w}{\partial p_i} - \frac{\partial w}{\partial q_i}\cdot\frac{\partial w}{\partial p_i}\{u, v\}\right]$$

$$= \{\{u, v\}, w\}$$

where we have used the results

$$\frac{\partial}{\partial q_i}\{u, v\} = \left\{u, \frac{\partial v}{\partial q_i}\right\} + \left\{\frac{\partial u}{\partial q_i}, v\right\}$$

$$\frac{\partial}{\partial p_i}\{u, v\} = \left\{u, \frac{\partial v}{\partial p_i}\right\} + \left\{\frac{\partial u}{\partial p_i}, v\right\}$$

which can be proved easily.

We next show that $Y = 0$. Indeed it is straightforward to see on expanding the Poisson brackets that

$$Y = \sum_i \sum_j \left[\frac{\partial^2 w}{\partial p_i \partial p_j}\left(\frac{\partial u}{\partial q_i}\frac{\partial u}{\partial q_j} - \frac{\partial u}{\partial q_i}\frac{\partial u}{\partial q_j}\right) + \frac{\partial^2 w}{\partial q_i \partial q_j}\left(\frac{\partial u}{\partial p_j}\frac{\partial u}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial p_j}\right)\right.$$

$$\left. + \frac{\partial^2 w}{\partial q_j \partial p_i}\left(-\frac{\partial v}{\partial q_i}\frac{\partial u}{\partial p_j} + \frac{\partial u}{\partial q_i}\frac{\partial u}{\partial p_j}\right) + \frac{\partial^2 w}{\partial q_i \partial p_j}\left(-\frac{\partial u}{\partial q_j}\frac{\partial u}{\partial p_i} + \frac{\partial v}{\partial q_j}\frac{\partial u}{\partial p_i}\right)\right]$$

On interchanging the indices i and j the first and second terms vanish due to anti-symmetry while the third term becomes equal to the fouth term except for a sign. We therefore conclude that $Y = 0$.

Hence

$$\{u, \{v, w\}\} + \{v, \{w, u\}\} = X = \{\{u, v\}, w\}$$

whence Jacobi identity follows.

V.5 Poisson Theorem : If corresponding to a given holonomic system u and v are two constants of motion then their Poisson bracket is also a constant of motion.

Proof : Using (V.12) we may unite

$$\frac{d}{dt}\{u,v\} = \frac{\partial}{\partial t}\{u,v\} + \{\{u,v\},H\}$$

Using the product rule and Jacobi identity we obtein on rearrangement

$$\frac{d}{dt}\{u,v\} = \left\{u,\frac{\partial u}{\partial t}\right\} + \left\{\frac{\partial u}{\partial t},v\right\} - \{\{v,H\},u\} - \{\{H,u\},v\}$$

$$= \left\{\frac{\partial u}{\partial t} + \{u,H\},v\right\} + \left\{u,\frac{\partial v}{\partial t} + \{v,H\}\right\}$$

$$= \left\{\frac{du}{dt},v\right\} + \left\{v,\frac{dv}{dt}\right\}$$

$$= 0$$

since $\frac{du}{dt} = 0$, $\frac{dv}{dt} = 0$, u and v being constants of motion.

Hence the theorem which is also known as Jacobi-Poisson theorem. Poisson theorem is useful in uncovering new constants of motion.

V. 6 Angular momentum : If l_x, l_y, l_z are the components of the angular momentum vector $\vec{l}$ then Poisson bracket of l_x and l_y may be obtained as

$$\{l_x,l_y\} = \{y\,p_z - z\,p_y,\ z\,p_x - x\,p_z\}$$

$$= \{yp_z, zp_x\} - \{yp_z, xp_z\} - \{zp_y, zp_x\} + \{zp_y, xp_z\}$$

$$= y\{p_z, z\}p_x + x\{z, p_z\}p_y$$

the other terms vanishing. Hence

$$\{l_x, l_y\} = l_z$$

Sililarly

$$\{l_y, l_z\} = l_x$$

$$\{l_z, l_x\} = l_y$$

It l_x and l_y are constants of motion then from Poisson's theorem it follows that their Poisson bracket namely l_z is also a constant of motion.

We summarize other results on the Poisson brackets involving l_x, l_y, l_z :

$$\{x, l_x\} = 0, \quad \{x, l_y\} = z \quad \{x, l_z\} = -y$$

$$\{y, l_x\} = -z, \quad \{y, l_y\} = 0, \quad \{y, l_z\} = x$$

$$\{z, l_x\} = y, \quad \{z, l_y\} = -x, \quad \{z, l_z\} = 0$$

If $\vec{n} = n_x \vec{i} + n_y \vec{j} + n_z \vec{k}$ be an arbitrary constant vector then

$$\left\{x, \vec{L}.\vec{n}\right\} = n_y z - n_z y = \left(\vec{n} \times \vec{r}\right)_x$$

$$\left\{y, \vec{L}.\vec{n}\right\} = n_z x - n_x z = \left(\vec{n} \times \vec{r}\right)_y$$

$$\left\{z, \vec{L}.\vec{n}\right\} = n_x y - n_y x = \left(\vec{n} \times \vec{r}\right)_z$$

The above results imply that

$$\left\{\vec{r}, \vec{L}.\vec{n}\right\} = \vec{n} \times \vec{r}. \qquad \text{(V.15)}$$

Similarly we can derive

$$\left\{\vec{p}, \vec{L}.\vec{n}\right\} = \vec{n} \times \vec{p} \qquad \text{(V.16)}$$

The transition from (V.15) to (V.16) involves two compensatory sign changes : changing $\vec{r}$ and $\vec{p}$ produces a change of sing not only in the difinition of the Poisson bracket but also in the definition of the angular momentum.

Problems :

(1) Show that $\left\{r^2, \vec{L}.\vec{n}\right\} = 0 \quad \left\{p^2, \vec{L}.\vec{n}\right\} = 0$

(2) Show that $\left\{\vec{r}.\vec{p}, \vec{L}.\vec{n}\right\} = 0$

(3) If $\vec{X}$ is defined as $\vec{X} = C_1 \vec{r} + C_2 \vec{p} + C_3\left(\vec{r} \times \vec{p}\right)$ where C_1, C_2, C_3 are arbitrary constants then show that $\left\{\vec{X}, \vec{L}.\vec{n}\right\} = \vec{n} \times \vec{X}$.

Ex. A paritcle of mass m is acted upon by a constant force F. Using the techniques of Poisson bracket show that x and p are in the forms

$$x = x_0 + \frac{p_0}{m}t + \frac{1}{2}\frac{F}{m}t$$

$$p = p_0 + Ft$$

which conform to the standard expressions

Let g be a function of the canonical variables (q, p)

A Taylor series expansion gives the representation

$$g = g_0 + \left(\frac{dg}{dt}\right)_0 t + \left(\frac{d^2g}{dt^2}\right)_0 \frac{t^2}{2!} + \dots$$

where g_0 is the initial value of g. Now

$$\frac{dg}{dt} = \{g, H\}$$

$$\frac{d^2g}{dt^2} = \left\{\frac{dg}{dt},\ H\right\} = \{\{g, H\}, H\}$$

and so on. As such

$$g = g_0 + \{g_0, H\}t + \left\{\{g_0, H\}, H\right\}\frac{t^2}{2!} + \dots$$

Now for the given problem we can write the underlying Hamiltonian as

$$H = \frac{p^2}{2m} - Fx$$

For $g = x$ we can immediately unite

$$\{x, H\} = \left\{x, \frac{p^2}{2m} - Fx\right\} = \frac{1}{2m}\left\{x, p^2\right\} = \frac{p}{m}\{x, p\} = \frac{p}{m}$$

$$\left\{\{x, H\},\ H\right\} = \left\{\frac{p}{m}, \frac{p^2}{2m} - Fx\right\} = -\frac{F}{m}\{p, x\} = \frac{F}{m}, \text{ a constant.}$$

So the series for $g = x$ terminates beyond $\left\{\{x, H\},\ H\right\}$ and we have

$$x = x_0 + \frac{p_0}{m}t + \frac{1}{2}\frac{F}{m}t^2$$

similarly choosing $g = p$ we would find $p = p_0 + Ft$.

Problem : Find the position x and momentum p at time t for the motion of a particle given by the Hamiltonian $H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$.

Answer : $x = x_0\left(1 - \frac{1}{2!}\omega^2 t^2 + \cdots\right) + \frac{p_0}{m\omega}\left(\omega t - \frac{1}{3!}\omega^3 t^3 + \cdots\right)$

$$= x_0 \cos \omega t + \frac{p_0}{m\omega} \sin \omega t$$

$$p = -m\omega\, x_0 \sin \omega t + p_0 \cos \omega t$$

Unit : VI ❑ Action Principles

VI. 1 The Principle of stationary action : Consider a physical system with n degrees of freedom. Its configuration is described by the n generalized coordinates $q_1, q_2, \ldots q_n$ representing the point P. P is called the representative point of the system. We can think of two points of view, the passive point and the active. In the passive point of view there are two observers sitting in the unprimed and primed frames respectively [see Figures VI. Ia and VI 1b].

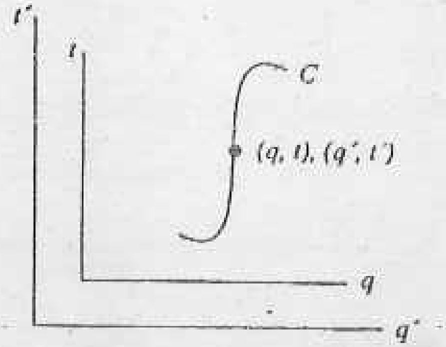


Fig VI.1a : Passive view

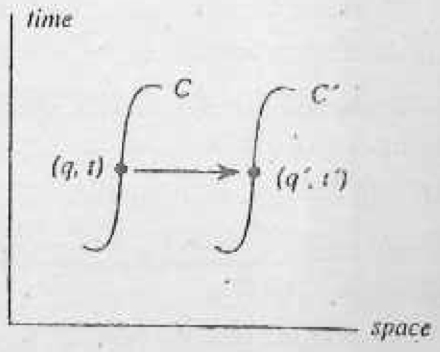


Fig VI. 1b : Active view

Each has his coordinate frame and assigns to the same physical point the coordinates (q, t) or (q', t') depending whether he is in the unprimed or primed frame. It is clear that the description of the trajectory C amounts to just a change of variables and nothing else. On the other hand, in the active view, the position of the observer is unchanged. So as the system evolves the points (q, t) are shifted to (q', t') resulting in the system paths C moving over to new C'.

What happens to the representative point P ?

As the system evolves, the coordinates $q_1, q_2, \ldots q_n$ change with time and P moves. As shown in figure VI.2

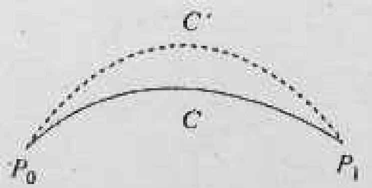


Fig VI.2 Representative points P_0 and P_1

let C be the actual path betweeen P_0 and P_1 which are points at times $t = t_0$ and t_1 respectively. The actual path is distinguished from the neighbouring ones, say C', in that along C equations of motion are always satisfied ; C is a dynamically allowed path. But paths such as C', which may even lie infinitesimally close to C, are handicapped in that these are only geometrically possible and dynamically impossible.

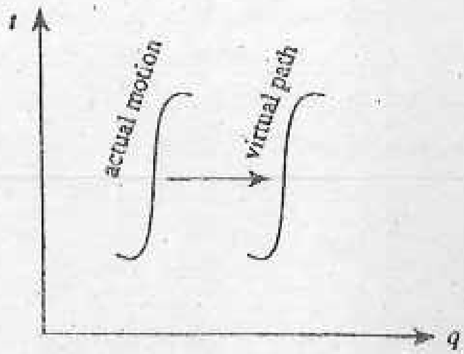


Fig. VI. 3 : Actual and virtual paths

We have already encountered the motion of virtual displacements δq_i which are consistent with the conditions of the constraints (see figure VI.3). Let us now imagine surrounding the actual path C by a family of neighbouring virtual paths C′ (see fig VI.2). Note that in a virtual displacement no passage of real time in involved.

To enquire into the rate of change of δq_i we see that

$$\frac{d}{dt}(\delta q_i) = \frac{d}{dt}(q_i + \delta q_i) - \frac{dq_i}{dt}$$

The *rhs* corresponds to the difference between the generalized velocity of the virtual path and generalized velocity on the actual path. In other words, it is the change in velocity as one steps from the actual path to the corresponding point on the neighbouring virtual (i.e. geometrically possible) curve. Thus

$$\frac{d}{dt}(\delta q_i) = \delta\left(\frac{dq_i}{dt}\right) = \delta \dot{q}_i \qquad \text{(VI.1)}$$

The joint action of the actual displacement d and the virtual one δ is thus commutative $d\delta = \delta d$.

We now look into the consequences of (VI.1) in the change in the Lagrangian. We find

$$\delta L = \sum_{i=1}^{n}\left[\frac{\partial L}{\partial q_i}\partial q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right]$$

$$= \sum_{i=1}^{n}\left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right]\delta q_i + \frac{d}{dt}\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\,\delta q_i$$

$$= \frac{d}{dt}\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\,\delta q_i \qquad \text{(VI.2)}$$

where we have used Lagrange's equations of motion which hold on the acutal trajectory. The apperance of total time derivative in the *rhs* of (IV.2) implies that if we integrate from an initial time t_0 to a later (final) one say t_1 then

$$\int_{t_0}^{t_1} \delta L\, dt = \sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\,\delta q_i\Bigg|_{t_0}^{t_1} \qquad \text{(VI.3)}$$

Now writing

$$S[q_i] = \int_{t_0}^{t_1} L(q_i, \dot{q}_i, t)\,dt \qquad \text{(VI.4)}$$

we immediately notice that δS corresponds to the lhs of (VI.3). This is because due to (VI.1) we can write $\delta S = \int \delta L\, dt + \int L\,\delta(dt) = \int \delta L\, dt + \int L\, d(\delta t) = \int \delta L\cdot dt$ for a virtual variation ($\delta t = 0$).

$S[q_i]$, which is a functional, is called the action of the path : it is the time integral of the Lagrangian between terminal values t_0 and t , along a particular path $q_i(t)$. We can represent (VI.3) as

$$\delta S = \sum_{i=1}^{n} p_i\,\delta q_i\Bigg|_{t_0}^{t_1} \qquad \text{(VI.5)}$$

Now if both the actual and virtual paths conicide at t_0 and t_1 which are initial and final times (see figure VI.2) respectively, the virtual displacements δq_i vanish at t_0 and t_1 implying from (VI.5)

$$\delta S = 0 \qquad \text{(VI.6)}$$

We are thus in a position to state the principle of stationary action (Hamilton's principle) which says that the actual path chosen by a physical system, between end points $P_0(q_0, t_0)$ and $P_1(q_1, t_1)$, is such that along it the action (VI.4) is stationary as compared with neighbouring virtual paths (i.e. which are geometrically possible) having the same terminal points (namely P_0 and P_1) as the actual trajectory.

We have so far exploited Lagrange's equations of motion to arive at Hamilton's principle of stationary action (VI.6) The converse also works : that is since

$$\delta S = \int_{t_0}^{t_1} \delta L \, dt = \int_{t_0}^{t_1} \sum_{i=1}^{n} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q_i \, dt$$

and δq_i are arbitraty independent variatious, $\delta S = 0$ provides Lagrange's equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \; i = 1, 2, \ldots n$$

oy equating to zero the coefficient of each δq_i.

The principle of stationary action is of fundamental importance in classical mechanics. It containly has an axiomatic status : as we have just observed, Lagrange's equations and the principle of stationary action are equivalent from an information content point of view for a physical system.

Problem : Show that $\int_{t_0}^{t_1} L(q, \dot{q}, t) dt$ and $\int_{t_0}^{t_1} \left[L(q, \dot{q}, t) + \frac{dF}{dt} \right] dt$

lead to the same equations of motion,.

$$S' = \int_{t_0}^{t_1} \left[L(q, \dot{q}, t) + \frac{dF}{dt} \right] d$$

$$= \int_{t_0}^{t_1} L(q, \dot{q}, t) dt + \left[F(t_1) - F(t_0) \right]$$

$$\therefore \; \delta S' = \delta S + \delta [F(t_1) - F(t_0)]$$

It is obvious that the second term in the rhs ought to vanish. Hence by the principle of stationary action $\delta S' = \delta S = 0$ and we are led to similar set of equations of motion.

We conclude this section by making a few remarks an the passive and active points of view. In the passive point of view, if C represents the path of an actual motion then the observer in the unprimed reference frame will write his action as

$$S[C] = \int_{t_0}^{t_1} L\left(q, \frac{dq}{dt}, t \right) dt$$

On the other hand, the observer in the primed reference frame will write his action as

$$S[C] = \int_{t_0}^{t_1} L'\left(q', \frac{dq'}{dt'}, t'\right)dt'.$$

Sine dt' is not expected to be equal to dt, i.e. $\frac{dt}{dt'} \neq 1$, the functional form of L and L' would, in general, be different. However, the form of the Lagrange's equations, obtained from the stationary character of S[C], one for the unprimed system and one for the primed system, would be similar. This is known as covariance :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \bar{q}}\right) = \frac{\partial L}{\partial q}$$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \bar{q}'}\right) = \frac{\partial L'}{\partial q'}, \quad \bar{q}' \equiv \frac{dq'}{dt'}$$

In the active point of view, for the actual path C, $S[C]$ is

$$S[C] = \int_{t_0}^{t_1} L\left(q, \frac{dq}{dt}, t\right)dt$$

It is to be stationary in comparison with other neighbouring paths. From Figure VI.2. C' is also an actual path. If we think of transformations which carry actual paths to actual paths, namely from C to C′, then such transformations are called invariance tranformations. Thus for invaniance the action for C′.

$$S[C'] = \int_{t_0}^{t_1} L\left(q', \frac{dq'}{dt'}, t'\right)dt'$$

should be stationary as compared with neighbouring paths to C′.

Here L remains the same.

VI.2 Corollaries :

(a) Hamilton's principle form D'Alembert's principle : The integral $\int_{t_0}^{t_1} (T - V)dt$ is stationary for an actual trajectory in comparision with neighbouring paths having coordinates of the end points fixed along with the terminal time instants.

Proof : From (II.19) D'Alembert's principle can be written as

$$\sum_{i=1}^{N}\left(\vec{F}_i^a - m_i \ddot{\vec{r}}_i\right)\cdot \delta \vec{r}_i = 0$$

Since $\frac{d}{dt}\left(\delta \vec{r}_i\right) = \delta \frac{d}{dt}\left(\vec{r}_i\right) = \delta \dot{\vec{r}}_i$, it can be also expressed in the from

$$\sum_{i=1}^{N} m_i\left[\frac{d}{dt}\left(\dot{\vec{r}}_i \,.\, \delta \vec{r}_i\right) - \dot{\vec{r}}_i \,.\, \delta \dot{\vec{r}}_i\right] = \sum_{i=1}^{N} \vec{F}_i^a \,.\, \delta \vec{r}_i$$

Using (III.2), (III.6), (III.8) and noting that $\sum m_i\left(\dot{\vec{r}}_i \,.\, \delta \dot{\vec{r}}_i\right) = \delta \sum \frac{1}{2} m_i\left(\dot{\vec{r}}_i^{\,2}\right)$

we are led to the result

$$\delta(T - V) = \frac{d}{dt}\left[\sum_{i=1}^{N} m_i\left(\dot{\vec{r}}_i \,.\, \delta \vec{r}_i\right)\right]$$

Integrating between t_0 and t_1 and since coordinates of the end points are fixed at t_0 and t_1 we get $\delta \int_{t_0}^{t_1} (T - V)dt = 0$ i.e. $\delta \int_{t_0}^{t_1} L\, dt = 0.$

(b) **Hamilton's canonical equations form the principle of stationary action :**

We can write

$$\int_{t_0}^{t_1} \delta L\, dt = \int_{t_0}^{t_1} \sum_{i=1}^{n} \delta\left(p_i\, \dot{q}_i - H\right)dt$$

using the definition of the Hamiltonian. Taking the variation

$$\int_{t_0}^{t_1} \delta L\, dt = \int_{t_0}^{t_1} \sum_{i=1}^{n}\left[p_i\, \delta\dot{q}_i + \left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right)\delta p_i - \frac{\partial H}{\partial q_i}\,\delta q_i\right]dt$$

But $\delta\dot{q}_i = \frac{d}{dt}(\delta q_i)$ and so

$$\int_{t_0}^{t_1} (p_i\, \delta\dot{q}_i)dt = \int_{t_0}^{t_1} p_i \frac{d}{dt}(\delta q_i)dt = -\int_{t_0}^{t_1} \dot{p}_i \,.\, \delta q_i\, dt$$

since δq_i vanishes at the terminal time points t_0 and t_1. Hence

$$\delta \int_{t_0}^{t_1} L\, dt = \int_{t_0}^{t_1} \delta L\, dt = \int_{t_0}^{t_1} \sum_{i=1}^{n}\left[-\left(\frac{\partial H}{\partial q_i} + \dot{p}_i\right)\delta q_i + \left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right)\delta p_i\right]dt$$

The equatities δq_i and δp_i being arbitrary and independent, the principle of stationary action $\delta \int_{t_0}^{t_1} L\, dt = 0$ gives

Hamilton's canonical equations :

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Ex. From Hamilton's principle write down the equations of the motion in spherical polar coordinates.

The Hamilton's principle is

$$\delta \int_{t_0}^{t_1} [T - V] dt = 0$$

where in spherical polar coordinates the kinetic energy T reads

$$T = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2\right), \quad (m = 1 \text{ has been set})$$

$$\therefore \quad \delta \int_{t_0}^{t_1} \left[\frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2\right) - V\right] dt = 0$$

Assuming $V = V(r, \theta, \phi)$ we get on taking the δ variation inside the integral

$$\int_{t_0}^{t_1} \left[\left\{\dot{r}\delta\dot{r} + r\dot{\theta}^2\delta r + r^2\dot{\theta}\delta\dot{\theta} + r\sin^2\theta\dot{\phi}^2\delta r + r^2\dot{\phi}^2 \sin\theta\cos\theta\delta\theta + r^2\sin^2\theta\dot{\phi}\delta\dot{\phi}\right\}\right.$$

$$\left. - \left(\frac{\partial V}{\partial r}\partial r + \frac{\partial V}{\partial \theta} + \frac{\partial V}{\partial \phi}\partial\phi\right)\right] dt = 0$$

But $\int_{t_0}^{t_1} \dot{r}\delta\dot{r} dt = -\int_{t_0}^{t_1} \ddot{r}\delta r\, dt$, $\int_{t_0}^{t_1} r^2\dot{\theta}\, \delta\dot{\theta} dt = -\int_{t_0}^{t_1} \frac{d}{dt}\left(r^2\dot{\theta}\right)\delta\theta\, dt$

$$\int_{t_0}^{t_1} r^2\sin^2\theta\, \dot{\phi}\, \frac{d}{dt}(\delta\phi) dt = -\int_{t_0}^{t_1} \frac{d}{dt}\left(r^2\sin^2\theta\dot{\phi}\right)\delta\dot{\phi}\, dt \quad \left(\because\ \delta\dot{\phi} = \frac{d}{dt}\delta\phi\right)$$

$$\therefore \int_{t_0}^{t_1} \left[\left(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\, \dot{\phi}^2 + \frac{\partial V}{\partial r}\right)\delta r\right.$$

$$\left. + \left\{\frac{d}{dt}\left(r^2\dot{\theta}\right) - r^2\sin\theta\cos\theta\dot{\phi}^2 + \frac{\partial V}{\partial \theta}\right\}\delta\theta + \left\{\frac{d}{dt}\left(r^2\sin^2\theta\dot{\phi}\right) + \frac{\partial V}{\partial \phi}\right\}\delta\phi\right] dt = 0$$

Since, δr, $\delta\theta$ and $\delta\phi$ are arbitrary and independent variations we get the equations of motion

$$\ddot{r} - r\dot{\theta}^2 - r^2 \sin^2\theta\dot{\phi}^2 = -\frac{\partial V}{\partial r}, \quad \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) - r\sin\theta\cos\theta\dot{\phi}^2 = -\frac{1}{r}\frac{\partial V}{\partial\theta},$$

$$\frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2\sin^2\theta\,\dot{\phi}\right) = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}$$

VI.3 Extended point transformation and Δ variation : There are certain physical situations in which moving boundaries are relevant. This is in contrast to the case of Hamilton's principle where the varied path shares with the actual path the same end pionts which are fixed in the sense $\delta q(t_0) = \delta q(t_1) = 0$ [see figure VI.2].

Curves with variable boundary points can appear as a result of an extended point transformation

$$q' = q'(q, t), \quad t' = t'(q, t)$$

involving both q and t. We restrict ourselves to the infinitesimal case

$$q' = q + \Delta q(q, t)$$

$$t' = t + \Delta t(q, t) \qquad \text{(VI.7)}$$

where Δq and Δt are infinitesimal small changes. Note that Δ is to be distinguished from δ in that we use δ for a virtual (time-frozen) change.

In the active point of view there is a single observer who observes the evolution of the system paths C to C' (see figure VI.1b). To him the difference in the action for two neghbouring paths appear as

$$\Delta S = S[C'] - S[C]$$

$$= \int_{t_0}^{t_1}\left[L\left(q', \frac{dq'}{dt'}, t'\right)dt' - L\left(q, \frac{dq}{dt}, t\right)\right]dt \qquad \text{(VI.8)}$$

where (q', t') are infintesimally different from (q, t) as in (VI.7).

It is to be emphasized that in the active view the observer remains unchanged. So, according to his coordinate frame of reference, system points (q, t) are observed to move to new positions (q', t'). This is why integration is done w.r.t. the t variable in (VI.8) between chosen terminal time points t_0 and t_1 of the active observer.

(VI.7) can be written as

$$dq' = dq + \left(\frac{d\Delta q}{dt}\right)dt$$

$$dt' = dt + \left(\frac{d\Delta t}{dt}\right)dt \qquad \text{(VI.9)}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q}\frac{\partial}{\partial q}$ as usual. Consequently

$$\frac{dq'}{dt'} = \frac{\left[\frac{dq}{dt} + \frac{d\Delta q}{dt}\right]dt}{\left[1 + \frac{d\Delta t}{dt}\right]dt}$$

$$= \left(1 + \frac{d\Delta t}{dt}\right)^{-1}\left(\frac{dq}{dt} + \frac{d\Delta q}{dt}\right)$$

$$\approx \left(\frac{dq}{dt} + \frac{d\Delta q}{dt}\right)\left(1 - \frac{d\Delta t}{dt}\right) \quad (\because\ \Delta t \text{ is infinitesimal})$$

$$\approx \left(1 - \frac{d\Delta t}{dt}\right)\frac{dq}{dt} + \frac{d\Delta q}{dt} \quad (\because\ \Delta q \text{ is infinitestimal}) \qquad \text{(VI.10)}$$

We therefore obtain the non-trivial result

$$\frac{dq'}{dt'} - \frac{dq}{dt} \neq \frac{d\,\Delta q}{dt}$$

i.e.. change in the generalized velocities is not the same as the time derivative of the change in the generalized coordinate.

Using (VI.9) and (VI.10), $L\left(q', \frac{dq'}{dt'}, t'\right)$ can be expanded as

$$L\left(q', \frac{dq'}{dt'}, t'\right)dt' = L\left[q + \Delta q, \left(1 - \frac{d\,\Delta t}{dt}\right)\frac{dq}{dt} + \frac{d\,\Delta q}{dt}, t + \Delta t\right]\left(1 + \frac{d\,\Delta t}{dt}\right)$$

$$= L\left[q + \Delta q, \frac{dq}{dt} + \left\{\frac{d\,\Delta q}{dt} - \left(\frac{d\,\Delta t}{dt}\right)\frac{dq}{dt}\right\}, t + \Delta t\right]\left(1 + \frac{d\,\Delta t}{dt}\right)$$

$$= \left[L\left(q, \frac{dq}{dt}, t\right) + \Delta q\frac{\partial L}{\partial q} + \left\{\frac{d\,\Delta q}{dt} - \left(\frac{d\,\Delta t}{dt}\right)\frac{dq}{dt}\right\}\frac{\partial L}{\partial \dot{q}} + \Delta t\frac{\partial L}{\partial t}\right]\left(1 + \frac{d\,\Delta t}{dt}\right)$$

$$= L\left(q, \frac{dq}{dt}, t\right) + \frac{\partial L}{\partial q}\Delta q + \frac{\partial L}{\partial \dot{q}}\left(\frac{d\,\Delta q}{dt} - \dot{q}\frac{d\,\Delta t}{dt}\right) + \frac{\partial L}{\partial t}\Delta t + L\frac{d\,\Delta t}{dt}$$

where we have kept only first order quantities in Δq, Δt and used Taylor expansion $\phi(x + \Delta x, y + \Delta y, z + \Delta z, \ldots) = \phi(x, y, z, \ldots) + \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y} + \Delta z \frac{\partial}{\partial z} + \ldots\right)$ $\phi(x, y, z, \ldots) + \ldots\ldots$

Thus (VI.8) becomes

$$\Delta S = \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial q} \Delta q + \frac{\partial L}{\partial \dot{q}} \left(\frac{d\, \Delta q}{dt} - \dot{q} \frac{d\, \Delta t}{dt} \right) + \frac{\partial L}{\partial t} \Delta t + L \frac{d\Delta t}{dt} \right] dt$$

Since on integration by parts we can write

$$\int \frac{\partial L}{\partial \dot{q}} \frac{d\, \Delta q}{dt} dt = \frac{\partial L}{\partial \dot{q}} \Delta q - \int \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \Delta q\, dt$$

$$\int \frac{\partial L}{\partial \dot{q}} \dot{q} \frac{d\, \Delta t}{dt} dt = \frac{\partial L}{\partial \dot{q}} \dot{q}\, \Delta t - \int \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \dot{q} \right) \Delta t\, dt$$

$$\int L \frac{d\, \Delta t}{dt} dt = L\, \Delta t - \int \frac{dL}{dt} \Delta t\, dt$$

the rhs can be rearranged as

$$\Delta S = \int_{t_0}^{t_1} dt \left[(\Delta q - \dot{q}\, \Delta t) \left\{ \frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right\} \right] + \left[\frac{\partial L}{\partial \dot{q}} \Delta q + \left(L - \frac{\partial L}{\partial \dot{q}} \dot{q} \right) \Delta t \right]_{t_0}^{t_1}$$

where we expressed the difference $\frac{dL}{dt} - \frac{\partial L}{\partial t} = \dot{q} \frac{\partial L}{\partial q}$.

For the actual path Lagrange's equations are satisfied and so the above reduces to

$$\Delta S = \left[\frac{\partial L}{\partial \dot{q}} \Delta q + \left(L - \frac{\partial L}{\partial \dot{q}} \dot{q} \right) \Delta t \right]_{t_0}^{t_1} \qquad \text{VI.11)}$$

Another way of displaying (VI.11) is to invoke the definition of the Hamiltonian and the canonical momentum p namely

$$\Delta S = [p\, \Delta q - H\, \Delta t]_{t_0}^{t_1} \qquad \text{(VI.12)}$$

For n number of particles, (VI.12) can be written as

$$\Delta S = \Delta \int_{t_0}^{t_1} L\, dt = \sum_i^n [p_i\, \Delta q_i - H\, \Delta t]_{t_0}^{t_1} \qquad \text{(VI.13)}$$

Let us restrict ourselves to the following propositions :

(i) H does not depend explicitly on t and so H is conserved.

(ii) H is conserved not only on the actual path but also on the varied path.

(iii) Varied paths are constrained such that Δq_i $(i = 1, 2, \ldots n)$ vanish at the end points but not Δt.

Under the above prescriptions (VI.13) simplifies to

$$\Delta S \equiv \Delta \int_{t_0}^{t_1} L\,dt = -H(\Delta t_1 - \Delta t_0) \tag{VI.14}$$

Now, the action integral reads

$$\int_{t_0}^{t_1} L\,dt = \int_{t_0}^{t_1} \left[\sum_{i1}^{n} p_i\,\dot{q}_i - H \right] dt = \int_{t_0}^{t_1} \Sigma p_i\,\dot{q}_i\,dt - H(t_1 - t_0) \tag{VI.15}$$

Taking Δ variation of (VI.15) we have

$$\Delta \int_{t_0}^{t_1} L\,dt = \Delta \int_{t_0}^{t_1} \sum_{i=1}^{n} p_i\,\dot{q}_i\,dt - H(\Delta t_1 - \Delta t_0) \tag{VI.16}$$

Comprision with (VI.14) yields

$$\Delta \int_{t_0}^{t_1} \sum_{i=1}^{n} p_i\,\dot{q}_i\,dt = 0 \tag{VI.17}$$

The integral in (VI.17) is generally referred in old books as the action or the action integral and (VI.17) as the principle of Least action. In recent times it is more customary to refer to the integral in Hamilton's principle as the actoin. Recall Hamilton's principle is the variational principle that states that the physical trejectory is the one for which the action is stationary.

If $L = T - V$ and $H = T + V$ then

$$\begin{aligned} \sum_{i=1}^{n} p_i\,\dot{q}_i &= H + L \\ &= T + V - (T - V) \\ &= 2T \end{aligned}$$

Hence another from of (VI.17) is

$$\Delta \int_{t_0}^{t_1} 2T\,dt = 0 \tag{VI.18}$$

Δ and δ variations : If the terminal time points be t_0 and t_1 then we already know that $\delta(t_0) = \delta(t_1) = 0$. Consider a function $f(t)$.

Then

$$\begin{aligned} f_1(t + \Delta t) - f(t) &= f_1(t) + \Delta t\ \dot{f}_1 - f(t) \\ &= [f_1(t) - f(t)] + \Delta t\ \dot{f}_1 \\ &= \delta f + \Delta t\ \dot{f}_1 \end{aligned}$$

Writing $f_1 = f + \Delta f$ we therefore obtain

$$\Delta f = \delta f + \Delta t\ \dot{f} \tag{VI.19}$$

where obviously $\Delta f = \delta f$ if $\Delta t = 0$. Differentiating (VI.19) w.r.t t we have

$$\frac{d}{dt}(\Delta f) = \frac{d}{dt}(\delta f) + \frac{d}{dt}(\Delta t)\dot{f} + \Delta t \frac{d}{dt}(\dot{f})$$

$$= \left[\delta\dot{f} + \Delta t \frac{d}{dt}(\dot{f})\right] + \frac{d}{dt}(\Delta t)\dot{f}$$

$$= \Delta\dot{f} + \frac{d}{dt}(\Delta t)\dot{f} \tag{VI.20}$$

where we have used $\Delta\dot{f} = \delta\dot{f} + \Delta t\ \ddot{f}$ analogous to (VI.19).

We thus arrive at the general formula

$$\frac{d}{dt}(\Delta f) = \Delta\dot{f} + \frac{d}{dt}(\Delta t)\dot{f} \tag{VI.21}$$

showing that $\frac{d}{dt}$ and Δ do not commute : $\frac{d}{dt}(\Delta) \neq \Delta \frac{d}{dt}$.

Parametric form of (VI. 18) :

Writing $\quad A = \int_{t_0}^{t_1} 2T\,dt \quad$ we have

$$A = \int_{t_0}^{t_1} \sqrt{2T}\sqrt{2T}\,dt = \int_{t_0}^{t_1} \sqrt{2(H - V)}\sqrt{\sum m\left(\frac{ds}{dt}\right)^2}\,dt$$

$$= \int_{t_0}^{t_1} \sqrt{2(H-V)}\sqrt{\Sigma\, m\, ds^2}$$

$$= \int_{P_0}^{P_1} \sqrt{2(H-V)}\sqrt{\Sigma\, m\left(\frac{ds}{d\lambda}\right)^2}\, d\lambda$$

where λ is an arbitrary parameter. Thus A is in the form

$$A = \int_{P_0}^{P_1} I\, d\lambda \tag{VI.22}$$

where $\quad I = \sqrt{2(H-V)}\sqrt{\Sigma\, m\left(\frac{ds}{d\lambda}\right)^2}$

which is a function of $(q_1, q_2, \ldots, q_n)$ and $\left(\frac{dq_1}{d\lambda}, \frac{dq_2}{d\lambda}, \ldots, \frac{dq_n}{d\lambda}\right)$ i.e. $(q_1, q_2, \ldots q_n)$ and $\left(q_1', q_2', \ldots q_n'\right)$, $q_r' \equiv \frac{dq_r}{d\lambda}$, $r = 1, 2, \ldots n$.

Proposition :

$$\frac{d}{d\lambda}\left(\frac{\partial I}{\partial q_r'}\right) = \frac{\partial I}{\partial q_r} \tag{VI.23}$$

where $r = 1, 2, \ldots\ n$

Proof : From (VI.18) i.e. $\Delta A = 0$ we have using (VI.19) and (VI.22)

$$0 = \delta \int_{P_0}^{P_1} I\, d\lambda \quad (\because\ I \text{ is independent of } t)$$

$$= \int_{P_0}^{P_1} \sum_{r=1}^{n} \left[\frac{\partial I}{\partial q_r}\delta q_r + \frac{\partial I}{\partial q_r'}\delta\, q_r'\right] d\lambda \tag{VI.24}$$

Now $\displaystyle\int_{P_0}^{P_1} \frac{\partial I}{\partial q_r'}\delta\, q_r'\, d\lambda = \int_{P_0}^{P_1} \frac{\partial I}{\partial q_r'}\frac{d}{d\lambda}(\delta\, q_r)\, d\lambda$

$$= \left[\frac{\partial I}{\partial q_r'}\delta q_r\right]_{P_0}^{P_1} - \int_{P_0}^{P_1} \frac{d}{d\lambda}\left(\frac{\partial I}{\partial q_r'}\right)\delta q_r\, d\lambda$$

$$= -\int_{P_0}^{P_1} \frac{d}{d\lambda}\left(\frac{\partial I}{\partial q_r'}\right)\delta q_r\, d\lambda \quad (\because\ \delta q_r = 0 \text{ at } P_0 \text{ and } P_1)$$

Subtituting the above result in (VI.24) we get

$$\int_{P_0}^{P_1} \sum_{r=1}^{n} \left[\frac{\partial I}{\partial q_r} - \frac{d}{d\lambda}\left(\frac{\partial I}{\partial q_r'} \right) \right] \delta q_r \, d\lambda = 0$$

Since δq_r are arbitrary and independent, (VI.23) follows.

Problem : A particle of unit mass is projected so that its total energy is h in a field of force whose potential is $\phi(r)$ at distance r from the origin. Deduce the differential equation of the path to be

$$C^2\left[r^2 + \left(\frac{dr}{d\theta}\right)^2 \right] = r^4[h - \phi(r)]$$

where C is a constant.

Here $V = -\frac{\mu}{r}$, $ds^2 = dr^2 + r^2 d\theta^2$, $H = h$, $U = \phi(r)$

$$\therefore \quad A = \int_{t_0}^{t_1} 2T \, dt$$

$$= \int_{t_0}^{t_1} \sqrt{2(H - V)} \sqrt{ds^2}$$

$$= \int_{p_0}^{p_1} \sqrt{2(h - \phi)} \sqrt{\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\theta}{d\lambda}\right)^2} \, d\lambda$$

where λ is a parameter other than t.

$$\therefore \quad I = \sqrt{2(h - \phi)}\sqrt{r'^2 + r^2\theta'^2}$$

where $r' = \frac{dr}{d\lambda}$, $\theta = \frac{d\theta}{d\lambda}$. Now θ is ignorable and so

$$\frac{d}{d\lambda}\left(\frac{\partial I}{\partial \theta'} \right) = 0 \text{ from (VI.23)}$$

$$\therefore \quad \frac{\partial I}{\partial \theta'} = \text{constant} = \sqrt{2}C \text{ (say)}$$

$$\therefore \quad \sqrt{2(h - \phi)} \, \frac{r^2\theta'}{\sqrt{r'^2 + r^2\theta'^2}} = \sqrt{2}C$$

On some rearrangement we get

$$C^2\left[r^2 + \left(\frac{dr}{d\theta}\right)^2 \right] = r^4[h - \phi(r)]$$

Unit : VII ❑ Symmetries and Constants of Motion

VII.1 Noether's theorem : In classical mechanics, Noether's theorem occupies a promiment position because according to this theorem if a symmetry is found to exist in a dynamical problem then there is a corresponding constant of motion.

Consider a rotation θ about the z-axis. The x and y coordinates transform according to

$$x' = x\cos\theta - y\sin\theta$$
$$y' = y\cos\theta + x\sin\theta$$

If θ be infinitesimal $x' = x - y\delta\theta,\ y' = y + x\,\delta\theta$. Accoridngly the Lagrangian $L \equiv L(x, y, z\ ;\ \dot{x}, \dot{y}, \dot{z}; t)$ changes as

$$L' \equiv L(x - y\delta\theta, y + x\delta\theta; z;\ \dot{x} - \dot{y}\,\delta\theta,\ \dot{y} + \dot{x}\delta\theta, \dot{z}\ ;\ t)$$

$$= L + \left[\left(x\frac{\partial L}{\partial y} - y\frac{\partial L}{\partial x}\right) + \left(\dot{x}\frac{\partial L}{\partial \dot{y}} - \dot{y}\frac{\partial L}{\partial \dot{x}}\right)\right]\delta\theta$$

where higher order terms are ignored. Defining the canonical momentum components to be

$$p_x \equiv \frac{\partial L}{\partial \dot{x}},\ p_y \equiv \frac{\partial L}{\partial \dot{y}},\ p_z \equiv \frac{\partial L}{\partial \dot{z}}$$

Lagrange's equations imply $\frac{d}{dt}p_x = \frac{\partial L}{\partial x},\ \frac{d}{dt}p_y = \frac{\partial L}{\partial y}$ and $\frac{d}{dt}p_z = \frac{\partial L}{\partial z}$.

Thus L' can be written as

$$L' = L + \frac{d}{dt}\left(x\,p_y - y\,p_x\right)\delta\theta$$

Invariance of the Lagrangian $L' = L$ leads to

$$\frac{d}{dt}\left(x\,p_y - yp_x\right) = \frac{d}{dt}l_z = 0$$

where l_z is the z-component of the angular momentum vector $\vec{l}$. Similarly performing rotations about the x and y-axis and seeking invariance of the Lagrangian yields constancy of the x and y components of $\vec{l}$ Considering all the three components together we conclude

$$\frac{d\vec{l}}{dt} = 0 \Leftrightarrow \text{rotational invariance of } L.$$

where $\vec{l} = \left(\vec{r} \times m\dot{\vec{r}}\right)$ is the total angular momentum.

VII. 2 Condition of invariance : It is interesting to define a reversible point transformation

$$q_i' = q_i' \;(q_1, q_2, \ldots, q_n \,;\, t) \qquad \text{(VII.1)}$$

with q_i' forming the Lagrangian $L'(q', \dot{q}', t)$ which is

$$L'(q', \dot{q}', t) = L[q(q', t), \dot{q}(q', \dot{q}', t), t] \qquad \text{(VII.2)}$$

Equations of motions are said to be invariant when these are same for the old and new variables coressponding to some suitable transformations of coordinates and velocities. Such transformations are called invariance transformations

For instane, as already noted by us in Unit III, same equations of motion are obtained if L' differs from L by a total time derivative, say $\frac{d\Lambda}{dt}$, where Λ depends on q' and t :

$$L'(q', \dot{q}', t) = L(q', \dot{q}', t) + \frac{d\Lambda(q', t)}{dt} \qquad \text{(VIII.3)}$$

The reason is that $\frac{d\Lambda}{dt}$ gives a vanishing contribution in Lagrangies equations of motion :

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}'}\left\{\frac{d\Lambda(q', t)}{dt}\right\}\right] = \frac{\partial}{\partial q'}\left\{\frac{d\Lambda(q', t)}{dt}\right\}$$

$$\text{or,} \quad \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}'}\left(\frac{\partial \Lambda}{\partial q'}\dot{q}' + \frac{\partial \Lambda}{\partial t}\right)\right] = \frac{\partial}{\partial q'}\frac{d\Lambda}{dt}$$

$$\text{or,} \quad \frac{d}{dt}\left(\frac{\partial \Lambda}{\partial q'}\right) = \frac{\partial}{\partial q'}\frac{d\Lambda}{dt}$$

which is consistent.

(VII.2) is the condition of invariance for the transformation defined by (VII. 1).

At the infinitesimal level we have, on combining (VII.2) and (VII.3) and keeping first order terms in δq and $\delta\dot{q}$.

$$L(q,\dot{q},t) = L(q+\delta q,\dot{q}+\delta\dot{q},t) + \frac{d}{dt}(\delta\Lambda)$$

$$= L(q,\dot{q},t) + \sum_{i=1}^{n}\left[\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i\right] + \frac{d}{dt}(\delta\Lambda)$$

i.e. $$\sum_{i=1}^{n}\left[\frac{\partial L}{\partial q_i}\delta q_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i\right] + \frac{d}{dt}(\delta\Lambda) = 0$$

Rearranging

$$\sum_{i=1}^{n}\left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right]\delta q_i + \frac{d}{dt}\left[\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\delta q_i + \delta\Lambda\right] = 0$$

where the first term may be dropped due to Lagranges' equation of motion leaving

$$\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\delta q_i + \delta\Lambda = \text{constant of motion} \qquad \text{(VII.4)}$$

Thus associated with an infinitesimal invariance transformation there is a constant of motion.

Indeed the constancy of angular momentum from the rotational invariance of the Lagrangian may be readily derived from (VII.4) for the general setting of a physical system with N particles. Now the lhs of (VII.4), for the rotation about z-axis, is

$$\sum_{i=1}^{N}\left[\frac{\partial L}{\partial \dot{x}_i}(-y_i\,\delta\theta) + \frac{\partial L}{\partial \dot{y}_i}(x_i\,\delta\theta)\right]$$

$$= \sum_{i=1}^{N}\left[m_i(-y_i\,\dot{x}_i + x_i\,\dot{y}_i)\delta\theta\right] = L_z\delta\theta$$

where we have considered the Lagrangian

$$L = \sum_{i=1}^{N}\frac{1}{2}m_i\left|\dot{\vec{r}}_i\right|^2 - \frac{1}{2}\sum_{i,j=1}^{N}V_{ij}\left(\left|\vec{r}_i - \vec{r}_j\right|\right)$$

for a closed system of N interacting particles guided by the potential V_{ij} between i-th and jth particle. Similarly we can consider infinitesimal rotations about the x and y-axis.

When we consider all the three components together

$$\vec{r}' - \vec{r} = \delta \vec{r} = \delta \vec{\theta} \times \vec{r} \text{ implying}$$

$$\sum_{i=1}^{N} \frac{\partial L}{\partial \dot{\vec{r}}_i} \cdot \delta \dot{\vec{r}}_i = \sum_{i=1}^{N} m_i \dot{\vec{r}}_i \cdot \delta \vec{\theta} \times \vec{r}_i$$

$$= \vec{l} \cdot \delta \vec{\theta}$$

where $\vec{l} = \sum_{i=1}^{N} r_i \times m_i \dot{\vec{r}}_i$ is the total angular momentum of the system.

Problem : Show that the constant of motion associated with the infinitesimal transformation for the spatial displacement leads to the constant of motion of the linear momentum.

Problem : Show that angular momentum is conserved for

$$L = \dot{\vec{r}} + \vec{r} \cdot \dot{\vec{r}} + \vec{r}^{\,2}$$

$$\vec{r} \to \delta \vec{\theta} \times \vec{r}$$

and $\quad \dot{\vec{r}} \to \delta \vec{\theta} \times \dot{\vec{r}}$

$$\delta L = \frac{\partial L}{\partial \vec{r}} \cdot \delta \vec{r} + \frac{\partial L}{\partial \dot{\vec{r}}} \cdot \delta \dot{\vec{r}}$$

$$= \left(2 \vec{r} + \dot{\vec{r}}\right) \cdot \left(\delta\theta \times \vec{r}\right) + \left(\vec{r} + 2 \dot{\vec{r}}\right) \cdot \left(\delta \vec{\theta} \times \dot{\vec{r}}\right)$$

$= 0$ [using the identity $\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \vec{b} \cdot \left(\vec{c} \times \vec{a}\right)$]

Ex. Is L invariant under translation ?

Ex. Consider the free particle problem described by the Lagrangian

$$L = \frac{1}{2} m\dot{x}^2$$

How does L transform under $x \to x + a(t)$. Since hem

$$x' = x + a(t) \text{ and } \dot{x}' = \dot{x} + \dot{a}.$$

$$L'(x', \dot{x}', t) = \frac{1}{2} m(\dot{x}' - \dot{a})^2$$

$$= L(x', \dot{x}', t) - m\dot{a}\dot{x}' + \frac{1}{2} m\dot{a}^2$$

If the transformation is an invariance then, as we know, there exists some function $\Lambda(x', t)$ such that

$$L'(x', \dot{x}', t) = L(x', \dot{x}', t) + \frac{d\Lambda}{dt}$$

from (VII.3). Hence comparing

$$\frac{\partial \Lambda}{\partial x'} = -m\dot{a}, \quad \frac{\partial \Lambda}{\partial t} = \frac{1}{2} m\dot{a}^2$$

i.e. $\ddot{a} = 0$. solving

$$a = C_1 + C_2 t$$

$$\Lambda = -mC_2 x' + \frac{1}{2} mC_2^2 t$$

where C_1 and C_2 are arbitrary constants. Hence x' is of the form $x' = x + C_1 + C_2 t$ which is a combination of spatial displacement ($C_2 = 0$) and Galilean transformation ($C_2 \neq 0$). We also observe that a transformation to a uniformly accelerating frame is not an invariance transformation.

Remark : Concerning invaniance under time displacement we have already considered the problem of moving boundary in connection with Δ variation in Unit VI. There we note that (VII.4) involves changes in the corodinate variables q_i which are dependent on the independnet variable t.

VII.3 Virial theorem : For a N-particle system at rest as a whole we define a quantity G as

$$G = \sum_{i=1}^{N} \vec{p}_i \cdot \vec{r}_i$$

where $\vec{r}_i$ denotes the position of a particle (i) with respect to a fixed origin. Clearly G is restricted to be finite since a material particle has neither an infinite momentum nor can it be infinitely away from the origin.

Translation of the origin by a finite amount say $\vec{r}_0$ transforms G according to

$$G' = \sum_{i=1}^{N} \vec{p}_i \cdot \left(\vec{r}_i - \vec{r}_0\right)$$

$$= G - \vec{r}_0 \sum_{i=1}^{N} \vec{p}_i$$

$$= G$$

from the conservation of linear momentum. Thus G is invariant under a shift of the origin.

The time rate of change of G is given by

$$\dot{G} = \sum_{i=1}^{N}\left(\dot{\vec{p}}_i . \vec{r}_i + \vec{p}_i . \dot{\vec{r}}_i\right)$$

$$= 2T + \sum_{i=1}^{N} \vec{F}_i . \vec{r}_i$$

$$= 2T + W$$

where T is the kinetic energy and W is the virial of the system. Like G, $W = \sum_{i=1}^{N} \vec{F}_i . \vec{r}_i$ too is independent of the choice of origin for the system at rest.

What is the time average of $2T + W$? We have

$$\Delta = \langle 2T + W \rangle = \lim_{\tau \to \infty} \frac{1}{\tau}\int_0^{\tau} \dot{G}\, dt$$

$$= \lim_{\tau \to \infty} \frac{G(\tau) - G(0)}{\tau}$$

$$= 0$$

since $G(\tau) - G(0)$ is always finite. Hence

$$\overline{T} = -\frac{1}{2}\overline{W}$$

This is the virial theorem.

Consider the particular case of the two-particle system :

$$W = \vec{f}_{12} . \vec{r}_2 + \vec{f}_{21} . \vec{r}_1 = \vec{f}_{12} . \left(\vec{r}_2 - \vec{r}_1\right)$$

where $\vec{f}_{ij}$, i, j = 1, 2, are the interparticle forces and we have used Newton's third law. If $V_{12} = \frac{-\gamma}{\left|\vec{r}_2 - \vec{r}_1\right|}$ is the interparticle potential (γ a constant) then $\vec{f}_{12}$ is

$$\vec{f}_{12} = -\frac{\gamma\left(\vec{r}_2 - \vec{r}_1\right)}{\left|\vec{r}_2 - \vec{r}_1\right|^3}$$

So

$$\vec{f}_{12}\cdot\left(\vec{r}_2-\vec{r}_1\right) = -\frac{\gamma}{\left|\vec{r}_2-\vec{r}_1\right|} = V_{12}$$

As such W represents the total potential energy V ($= V_{12}$) of the system leading to

$$\bar{T} = -\frac{1}{2}\bar{W} = -\frac{1}{2}\bar{V}$$

From the conservation of total energy we write the above result as

$$E = -\bar{T} = \frac{1}{2}\bar{V}$$

VII.4 Brachistochrone problem : This problem is concerned with finding a plane curve that joins two points such that the time required for a particle faling under gravity along it, from an upper point to the lower, is a minimum.

Let x-axis be horizontal and z-axis be vertically upwards. P and Q are initial and final positions of the particle respectively. If the velocity and height at P is (v, z) and the same at Q is (v_f, z_f) then

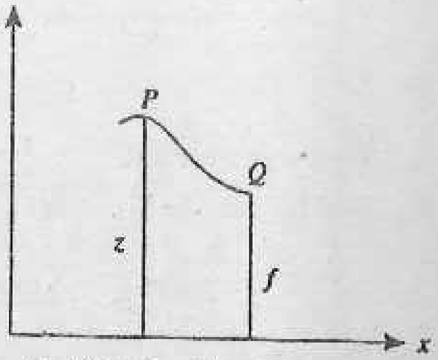


Fig. VII. 1 Brachistochrone curve

Total energy at $P = \frac{1}{2}mv^2 + mg\,z$

Total energy at $Q = \frac{1}{2}mv_f^2 + mg\,z_f$

By conservation of energy these are equal so that

$$v^2 = v_f^2 + 2g\left(z_f - z\right)$$

Now $v = \frac{ds}{dt}$ for an element of arc ds of PQ in time dt. Hence the above equation can be expressed as

$$t = \int_P^Q ds\left[v_f^2 + 2g\left(z_f - z\right)\right]^{-1/2}$$

$$= \int_{z}^{z_0} \frac{\left[1+\left(\frac{dx}{dz}\right)^2\right]^{1/2}}{\left[v_f^2 + 2g(z_f - z)\right]^{1/2}} dz \tag{VII.4}$$

where we have put $ds = \sqrt{dx^2 + dz^2}$

(VII.4) has the same form of an action with x playing the role of "generalized coordinate", z that of "time" and the integrand that of the "Lagrangian L" :

$$L = \frac{\left[1+\left(\frac{dx}{dz}\right)^2\right]^{1/2}}{\left[v_f^2 + 2g(z_f - z)\right]^{1/2}}$$

$$= \frac{(1+x'^2)^{1/2}}{\left[v_f^2 + 2g(z_f - z)\right]^{1/2}}, \quad x' = \frac{dx}{dz}$$

$$= L(x, x', z), \quad x' = \frac{dx}{dz}$$

Note x is ignorable.

Brachistochrone is the path of minimum time obtainable from

$$\delta \int_{z}^{z_0} L(x, x', z) dz = 0$$

This yields

$$\frac{d}{dz}\left(\frac{\partial L}{\partial x'}\right) = \frac{\partial L}{\partial x} = 0$$

from which

$$\frac{\partial L}{\partial x'} = \text{constant}$$

i.e.

$$\frac{x'}{(1+x'^2)^{1/2}\left[v_f^2 + 2g(z_f - z)\right]^{1/2}} = C \text{ (say)}$$

$$\text{or,} \quad x'^2(1+x'^2)^{-1}\left[v_f^2 + 2g(z_f - z)\right]^{-1} = C^2$$

This gives

$$\frac{1}{x'^2} = \frac{1 - C^2\left[v_f^2 + 2g(z_f - z)\right]}{C^2\left[v_f^2 + 2g(z_f - z)\right]}$$

i.e. $$\left(\frac{dx}{dz}\right)^2 = \frac{b - z}{a + z}$$

where $a = \left(1 - C^2v_f^2 - 2gC^2z_f\right)/2gC^2$ and $b = \left(v_f^2 + 2g\,z_f\right)/2g$. Setting $z = -\left(\frac{a-b}{2}\right) - \left(\frac{a+b}{2}\right)\cos\theta$ facilitates integration since we have

$$dx = \int \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\left(\frac{a+b}{2}\sin\theta\right)d\theta$$

implying

$$x = \frac{a+b}{2}(\theta - \sin\theta) + \text{constant}$$

Thus we get the curve for minimum time, the "brachistochrone", in parametric form with x and z appearing as functions of θ. It is a cycloid.

VII.5. Other examples :

(a) Law of reflection : Consider a ray of light travelling from the point P_0 to P_1 through the point of reflection at $(x, 0)$ on a mirror M. The time taken is (see Fig VII.2)

$$t = \frac{1}{c}\sqrt{(x - x_0)^2 + y_0^2} + \frac{1}{c}\sqrt{(x_1 - x)^2 + y_1^2}$$

c being the velocity of light. To minimize t we require $\frac{\partial t}{\partial x} = 0$.

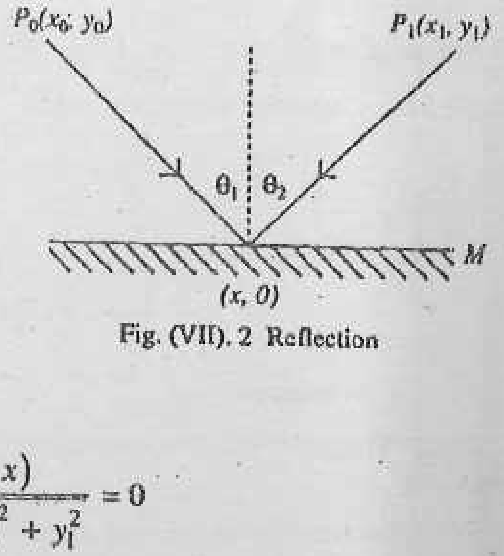


Fig. (VII). 2 Reflection

It gives

$$\frac{(x - x_0)}{\sqrt{(x - x_0)^2 + y^2}} - \frac{(x_1 - x)}{\sqrt{(x_1 - x)^2 + y_1^2}} = 0$$

i.e. $\sin\phi_0 = \sin\phi_1$

$\Rightarrow \quad \phi_0 = \phi_1 \quad$ (Law of reflection)

(b) **Law of refraction :** Consider two mediums S_0 and S_1 characterized by index of refraction n_0 and index of refraction n_1. For the light travelling from P_0 to P_1 through the point $(x, 0)$ the time taken is (See Fig.VII.3)

$$t = \frac{n_0}{c}\sqrt{(x - x_0)^2 + y_0^2} + \frac{n_1}{c}\sqrt{(x_1 - x)^2 + y_1^2}$$

Minimization of t requries $\frac{\partial t}{\partial x} = 0$ which gives

$$n_0 \sin\phi_0 = n_1 \sin\phi_1$$

This is Snell's law of refraction.

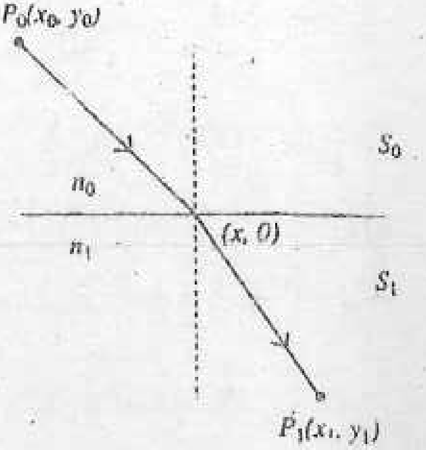


Fig. VIII. 3 Refraction

(c) Show that the shortest distance between two points in a plane is a straight line

Since $\quad ds = \sqrt{dx^2 + dy^2}$ we have

$$s = \int_{x_0}^{x_1}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_{x_0}^{x_1} L(y, y', x)dx$$

where $y' = \frac{dy}{dx}$ and $L = \left(1 + y'^2\right)^{1/2}$. It is in a typical "action" with x playing the role of time. Lagrange's equation reads

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial y} = 0$$

since y is ignorable. So

$$\frac{y'}{\sqrt{1 + y'^2}} = \text{constant}$$

i.e. $\quad y' = \text{constant}$.

Integrating we get the equation of the straight line $y = mx + c$, m and c are constants.

(d) Show that the shortest distance between two points on the surface of a sphere is a great circle.

Here $\quad ds = \sqrt{a^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)}$

a being the radius of the sphere.

$$\therefore\ s = \int_{\theta_0}^{\theta_1} a\left[1 + \sin^2\theta(\phi')^2\right]^{1/2} d\theta$$

where $\phi' = \frac{d\phi}{d\theta}$. Again it is of the form $s = \int_{\theta_0}^{\theta_1} L(\phi, \phi', \theta)d\theta$ with θ playing the role of time. The associated Lagrange's equation gives

$$\frac{d}{d\theta}\left(\frac{\partial L}{\partial \phi'}\right) - \frac{\partial L}{\partial \phi} = 0$$

ϕ being ignorable and $L = a\left[1 + \sin^2\theta(\phi')^2\right]^{1/2}$. We therefore have

$$\frac{a\sin^2\theta\,\phi'}{\left[1 + \sin^2\theta\,\phi'^2\right]^{1/2}} = \text{constant} = \lambda\ (\text{say})$$

$$\therefore\quad d\phi = C(a^2\sin^4\theta - C^2\sin^2\theta)^{-1/2}d\theta$$

Integrating

$$\phi + \mu = C\int \frac{d\theta}{\sqrt{a^2\sin^4\theta - c^2\sin^2\theta}},\ \mu \text{ a constant}$$

Put $t = C\cot\theta$ or, $dt = -\operatorname{cosec}^2\theta\,d\theta$. We get

$$\phi + \mu = C\int \frac{-\sin^2\theta\,dt}{\sin\theta\sqrt{a^2\sin^2\theta - C^2}}$$

$$= -C\int \frac{\sin\theta\,dt}{\sqrt{a^2\sin^2\theta - C^2}}$$

$$= \int \frac{dt}{\sqrt{\left(a^2 - C^2\right) - t^2}}$$

$$= \sin^{-1}\frac{t}{b},\quad b^2 = a^2 - c^2$$

$$\therefore\ C\cot\theta = b\sin(\phi + \mu)$$

or, $C\cos\theta = b\sin\theta\sin\phi\cos\mu + b\sin\theta\cos\phi\sin\mu$

This is in the form $z = px + qy$, $p = \sin\mu$ and $q = \cos\mu$, which is the equation of a plane passing through the origin.

(c) Find the equation of the curve which makes the surface area of revolution generated by rotating the curve $y = y(x)$ around the x-axis.

Area element of surface of revoution is (see Fig. VII.4)

$$dA = 2\pi y\, ds$$

$$= 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\therefore \quad A = 2\pi \int_{x_0}^{x_1} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_{x_0}^{x_1} L\,dx$$

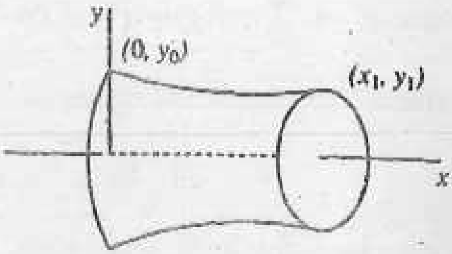


Fig. VII. 4 Surface area of revolution

where $L = 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ with x playing the role of time. Note that x is absent from the Lagrangian implying that the associated Hamiltonian

$$H = \frac{\partial L}{\partial y'}y' - L, \quad y' = \frac{dy}{dx}$$

$$= \text{constant}$$

As such

$$2\pi y\frac{1}{2}\frac{2y'^2}{\sqrt{1+y'^2}} - 2\pi y\sqrt{1+y'^2} = \text{constant}$$

i.e. $$\frac{-2\pi y}{\sqrt{1+y'^2}} = \text{constant}$$

Let $\frac{y}{\sqrt{1+y'^2}} = C$, a constant. It corresponds to the minimum value of y where $\frac{dy}{dx} = 0$.

$$\therefore \quad \frac{dy}{dx} = \frac{1}{C}\sqrt{y^2 - C^2}$$

which yields the equation of the catenary on integration

$$y = C\cos h\left[\left(x - x_0\right)/C\right]$$

where x_0 is a constant.

Unit : VIII ☐ The Theory of Canonical Transformations

VII.1 Introduction : Le us begin by asking the question whether a transformation from the canonical variables (q, p) to a new set (Q, P) is feasible such that Hamilton's equations in the form

$$\dot{Q} = \frac{\partial K}{\partial P}, \quad \dot{P} = -\frac{\partial K}{\partial Q} \tag{VIII.1}$$

hold where $K = K(Q, P) = H(q(Q, P), p(Q, P))$ is the transformed H in terms of the new variables Q and P. It is obvious that this will not be generally so except for some special cases. Such restricted transformations for which (VIII. I) holds are called canomical transformations implying that the new variables (Q, P) too form a canonical set.

Writing $Q = Q(q, p)$, $P = P(q, p)$ it is thus clear that these ought to satisfy Hamilton's canonical equations namely

$$\begin{aligned} \frac{dQ}{dt} &= \{Q, H\}_{(q,p)} = \frac{\partial Q}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial H}{\partial q} \\ \frac{dP}{dt} &= \{P, H\}_{(q,p)} = \frac{\partial P}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial H}{\partial q} \end{aligned} \tag{VIII.2}$$

Further

$$\begin{aligned} \frac{\partial H}{\partial q} &= \frac{\partial K}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial K}{\partial p}\frac{\partial P}{\partial q} \\ \frac{\partial H}{\partial p} &= \frac{\partial K}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial K}{\partial P}\frac{\partial P}{\partial p} \end{aligned} \tag{VIII.3}$$

Substituting (VIII.3) in (VIII.2) we find

$$\begin{aligned} \frac{dQ}{dt} &= \frac{\partial Q}{\partial q}\left(\frac{\partial K}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial K}{\partial P}\frac{\partial P}{\partial p}\right) - \frac{\partial Q}{\partial P}\left(\frac{\partial K}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial K}{\partial P}\frac{\partial P}{\partial q}\right) \\ &= \frac{\partial K}{\partial P}\left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right) \\ &= \frac{\partial K}{\partial P}\{Q, P\}_{(q,p)} \\ \frac{dp}{dt} &= -\frac{\partial K}{\partial Q}\{Q, P\}_{(q,p)} \end{aligned} \tag{VIII.4}$$

(VIII.4) suggests that (Q, P) satisfy Hamilton's canonical equations provided $\{Q, P\}_{(q, p)}$ is a constant. Taking the constant to be unity without any loss of generalty, the canonical transformation may be defined to be a transformation for which

$$\{Q, P\}_{(q, p)} = 1 \tag{VIII.8}$$

Formally the condition for the canonical transformation $(q, p) \rightarrow (Q, P)$ is given by

$$\left\{Q_i, Q_i\right\}_{(q,p)} = 0, \left\{P_i, P_j\right\}_{(q,p)} = 0, \left\{Q_i, P_j\right\}_{(q,p)} = \delta_{ij} \tag{VIII.9}$$

For the one-dimensional space, the Poisson bracket $\{Q,P\}_{(q,p)}$ is essentially equal to the Jacobian determinant :

$$\begin{aligned}\{Q, P\}_{(q,p)} &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \\ &= \frac{\partial(Q, P)}{\partial(q, p)}\end{aligned} \tag{VIII.10}$$

Conversely

$$\{q, p\}_{(Q,P)} = \frac{\partial(q, p)}{\partial(Q, P)} = \left[\frac{\partial(Q, P)}{\partial(q, p)}\right]^{-1} = \left[\{Q, P\}_{(q,p)}\right]^{-1} \tag{VIII.11}$$

Consider the one-dimensional phase space (q, p) and D a region in it bounded by a closed curve A. For a canonical transformation

$$\begin{aligned}\int_D dQ\, dP &= \int_D \frac{\partial(Q, P)}{\partial(q, p)}\, dq\, dp \\ &= \int_D \{Q, P\}_{(q,p)}\, dq\, dp \\ &= \int_D dq\, dp\end{aligned}$$

$$\oint_A (p\, dq - P\, dQ) = 0$$

Here the quantity $(p\, dq - pdQ)$ must be a perfect differential. Indeed expressing

$$\begin{aligned}p\, dq - P\, dQ &= p\, dq - P\left(\frac{\partial Q}{\partial q} dq + \frac{\partial Q}{\partial p} dp\right) \\ &= \left(p - P\frac{\partial Q}{\partial q}\right)dq - P\frac{\partial Q}{\partial p}\, dp\end{aligned}$$

the condition for a perfect differential is

$$\frac{\partial}{\partial p}\left(p - p\frac{\partial Q}{\partial q}\right) = \frac{\partial}{\partial q}\left(-p\frac{\partial Q}{\partial p}\right)$$

which works out to $\{Q, P\}_{(q,p)} = 1$

Putting $pdq - PdQ = dG_1$, we call G_1 to be the generating function of the transformation $(q, p) \to (Q, P)$. It is a function of q and Q [i.e. $G_1 = G(q, Q)$] and we have straightforwardly

$$p = \frac{\partial G_1}{\partial q},\ P = -\frac{\partial G_1}{\partial Q} \qquad \text{(VIII.12)}$$

Ex. : Consider the transformation $(q, p) \to (Q, P)$ given by

$$q = \sqrt{\frac{2P}{m\omega}}\sin\theta,\quad p = \sqrt{2m\omega P}\cos\theta$$

Inverting

$$Q = \tan^{-1}\left(m\omega\frac{q}{p}\right),\ P = \frac{1}{2\omega}\left(\frac{p^2}{m} + m\omega^2 q^2\right)$$

$$\therefore\quad \frac{\partial Q}{\partial q} = \frac{m\omega p}{p^2 + m^2\omega^2 q^2},\ \frac{\partial Q}{\partial p} = -\frac{m\omega q}{p^2 + m^2\omega^2 q^2}$$

$$\{Q, P\}_{(q,p)} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}$$

$$= \frac{1}{p^2 + m^2\omega^2 q^2}\left(m\omega p\frac{\partial P}{\partial p} + m\omega q\frac{\partial P}{\partial q}\right)$$

Since $\frac{\partial P}{\partial p} = \frac{p}{\omega m},\ \frac{\partial P}{\partial q} = m\omega q$ it follows that $\{Q, P\}_{(q,p)} = 1$. Hence, the transformation is canonical.

Next we check whether $pdq - PdQ$ is a perfect differential For the generating function G_1, q and Q are independent variables. Expressing p and P in terms of these namely $p = m\omega q\cot\theta$, $P = \frac{1}{2}m\omega q^2\operatorname{cosec}^2\theta$ we see that

$$pdq - pdQ = d\left(\frac{1}{2}m\omega q^2\cot\theta\right)$$

which is indeed an exact differential. Hence G_1 for this problem is $G_1(q, Q) = \frac{1}{2}m\omega q^2\cot\theta$.

Applying the above transformation to the specific case of the harmonic oscillator described by the Hamiltonian $H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2$, it is clear from the above form of P that $H \to K = P\omega$ The accompanying Hamilton's equations are

$$\dot{Q} = \frac{\partial K}{\partial P} = \omega$$

$$\dot{P} = -\frac{\partial K}{\partial Q} = 0$$

Solving for Q and P we find $Q = \omega t + t_0$, $P = b$ where t_0 and b are constants of integration. Switching to the original variables we get

$$q = \sqrt{\frac{2b}{m\omega}}\sin\left(\omega t + t_0\right)$$

$$p = \sqrt{2m\omega b}\cos\left(\omega t + t_0\right)$$

which conform to the standard forms.

The above example serves to illustrate the utility of canonical transformatoin in that it is often possible to adopt new set of canonical variables which simplifies the basic equations a great deal thus facilitating generation of the solutions which are otherwise very complicated to determine.

Problems : (1) Show that $Q = -p$, $P = q + \lambda p^2$ is a canonical transformatoin where λ is a constant.

(2) Show that $Q = q\cos\theta - \frac{p}{m\omega}\sin\theta$, $P = m\omega q\sin\theta + p\cos\theta$ is a canonical transformation.

VIII.2. Types of Canonical transfromations :

As we have already noted in Unit V, the Lagrangian—Hamiltonian relationship for a system with n degrees of freedom, is provided by the Legendre transformation

$$L = \sum_{i=1}^{n} p_i\dot{q}_i - H(q, p, t)$$

Indeed it results in the cononical equations $\dot{q} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial q}$. It is therefore suggestive that to generate a similar set of equations, namely (VIII.1), under the canonical transformation $(q, p) \to (Q, P)$, the quantity that is relevant for the new

Lagrangian should be $\sum_{i=1}^{n} P_i \dot{Q}_i - K(Q, P, t)$. However, we have also learnt in Unit III that the addition of a total derivative term to a Lagrangian does not affect the equations of motion. So we define a canonical transformation $(q, p) \to (Q, P)$ as the one guided by the following restriction :

$$\sum_{i=1}^{n} p_i \dot{q}_i - H = \sum_{i=1}^{n} P_i \dot{Q}_i - K + \frac{dG_1}{dt} \qquad \text{(VIII.13)}$$

G_1 being the generating function of the transformation. The above form is equivalent to

$$\sum_{i=1}^{n} p_i \, dq_i - H \, dt = \sum_{i=1}^{n} P_i \, dQ_i - K \, dt + dG_1 \qquad \text{(VIII.14)}$$

with $G_1 = G_1(q, Q, t)$

Of the $4n$ variables (q_i, p_i) and (Q_i, P_i), validity of (VIII.1) ensures that only 2n of them are independent. A few different types of canonical transformations are focussed below.

Type 1 Canonical transformation : Type 1 canonical transformation corresponds to treating q_i and Q_i as independent. Naturally

$$dG_1 = \sum_{i=1}^{n} \left(\frac{\partial G_1}{\partial q_i} dq_i + \frac{\partial G_1}{\partial p_i} dp_i \right) + \frac{\partial G_1}{\partial t} dt$$

Putting this representation of dG_1 in (VIII.14) we deduce

$$p_i = \frac{\partial G_1}{\partial q_i}(q, Q, t), \quad P_i = -\frac{\partial G_1(q, Q, t)}{\partial Q_i} \qquad \text{(VIII.15)}$$

which are identical to (VIII.12) along with

$$K = H + \frac{\partial G_1}{\partial t} \qquad \text{(VIII.16)}$$

The first of (VIII.15) gives Q_i in terms of q_i and p_i which when substituted in the second of (VIII.15) yields P_i (of course we assume $\det \left| \frac{\partial^2 G_1}{\partial q_i \, dQ_j} \right| \neq 0$) . K is the new Hamiltonian.

Type 2 Canonical transformation :

In Type 2 canonical transformation the independent variables are taken to be (q_i, P_i). This case presents no problem to deal with since we can express

$$\sum_{i=1}^{n} P_i\, dQ_i = d\left(\sum_{i=1}^{n} P_i\, Q_i\right) - \sum_{i=1}^{n} Q_i\, dP_i$$

resulting in

$$\sum p_i\, dq_i + \sum_{i=1}^{n} Q_i\, dP_i - H\, dt = d\left(\sum_{i=1}^{n} P_i\, Q_i\right) - K\, dt + dG_1 \qquad \text{(VIII.17)}$$

Defining the accompanying generating function to be G_2 we have

$$G_2 = G_1 + \sum_{i=1}^{n} P_i\, Q_i \qquad \text{(VIII.18)}$$

Viewed as function of q, P and t we write

$$dG_2(q, P, t) = \sum_{i=1}^{n}\left(\frac{\partial G_2}{\partial q_i}\, dq_i + \frac{\partial G_2}{\partial P_i}\, dP_i\right) + \frac{\partial G_2}{\partial t}\, dt$$

implying from (VIII.17)

$$p_i = \frac{\partial G_2(q, p, t)}{\partial q_i},\ Q_i \frac{\partial G_2(q, P, t)}{\partial P_i} \qquad \text{(VIII.19)}$$

along with

$$K = H + \frac{\partial G_2}{\partial t} \qquad \text{(VIII.20)}$$

Here the first of (VIII.19) gives P_i in terms of q_i and p_i which when substituted in the second of (VIII.19) yields Q_i (we assume det$\left|\frac{\partial^2 G_2}{\partial q_i\, \partial P_j}\right| \neq 0$). K as given by (VIII.20) is the new Hamiltonian.

The other types of canonical transformations are the Type 3 canonical transformation and Type 4 canonical transformation. In Type 3 canonical transformation the underlying generating function G_3 is a function of the independent variables p_i, Q_i and t and defined as

$$G_3(p, Q, t) = G_1 - \sum_{i=1}^{n} q_i\, p_i \qquad \text{(VIII.21)}$$

It leads to

$$q_i = -\frac{\partial G_3}{\partial p_i},\ P_i = -\frac{\partial G_3}{\partial Q_i},\ K = H + \frac{\partial G_3}{\partial t} \qquad \text{(VIII.22)}$$

In the Type 4 canonical transfromation the generating function is given by

$$G_4(p, P, t) = G_1 + \sum_{i=1}^{n} Q_i P_i - \sum_{i=1}^{n} q_i p_i \qquad \text{(VIII.23)}$$

in which p_i and P_i are treated as independent variables. We then have

$$q_i = -\frac{\partial G_4}{\partial p_i},\ Q_i = \frac{\partial G_4}{\partial P_i},\ K = H + \frac{\partial G_4}{\partial t} \qquad \text{(VIII.24)}$$

Note that the relationships (VIII.22) and (VIII.24) are subjected to

$$\det\left|\frac{\partial^3 G_3}{\partial p_i\,\partial Q_i}\right| \neq 0 \text{ and } \det\left|\frac{\partial^2 G_4}{\partial p_i\,\partial P_i}\right| \neq 0.$$

Ex. 1. Consider the transformation

$$Q = -p,\quad P = q + \lambda p^2$$

where λ is a constant. By the Poisson bracket test namely $[Q, P]_{(q,\,p)} = 1$ we conclude that the transformation is a canonical transformation.

The Type 1 generating function is obtained by showing that $pdq - PdQ$ is a perfect differential :

$$pdq - PdQ = (-Q)dq - (q + \lambda Q^2)dQ$$

$$= d(-qQ - \tfrac{1}{3}\lambda Q^3)$$

Thus

$$G_1(q, Q) = -qQ - \tfrac{1}{3}\lambda Q^3$$

On the other hand the Type 2 generating function is obtained by treating q and P as independent variables.

From (VIII.18) we have

$$G_2(q, p) = G_1 + PQ$$

$$= -qQ - \tfrac{1}{3}\lambda Q^3 + \left(q + \lambda Q^2\right)Q$$

$$= \frac{2}{3}\lambda Q^3$$

$$= \frac{2}{3}\lambda\left(\frac{P-q}{\lambda}\right)^{3/2}$$

Check that

$$\left(\frac{\partial G_2}{\partial q}\right)_P = -\frac{1}{\sqrt{\lambda}}(P-q)^{1/2} = p$$

$$\left(\frac{\partial G_2}{\partial P}\right)_q = \frac{1}{\sqrt{\lambda}}(P-q)^{1/2} = Q$$

which are as required.

Ex. 2.

The transformation

$$Q = q\cos\theta - \frac{p}{m\omega}\sin\theta$$

$$P = m\omega q\sin\theta + p\cos\theta$$

is easily seen to be canonical due to $[Q, P]_{(q,p)} = 1$. We also find

$$p = m\omega\left(q\cot\theta - \frac{Q}{\sin\theta}\right)$$

$$P = m\omega\left(\frac{q}{\sin\theta} - Q\cot\theta\right)$$

Hence the quantity $pdq - PdQ$ can be expressed as

$$pdq - PdQ = d\left[\frac{1}{2}m\omega\left(q^2 + Q^2\right)\cot\theta - m\omega qQ\operatorname{cosec}\theta\right]$$

The Type 1 generating function $G_1(q, Q)$ may therefore be identified as

$$G_1(q, Q) = \frac{1}{2}m\omega\left(q^2 + Q^2\right)\cot\theta - m\omega\, qQ\operatorname{cosec}\theta$$

The Type 2 generating function can be obtained from $G_2 = G_1 + PQ$. We get

$$G_2 = \frac{1}{2}m\omega\left(q^2 - Q^2\right)\cot\theta$$

To assign the right variable dependence on G_2 namely q and P we note that

$$Q = \frac{q}{\cos\theta} - \frac{P}{m\omega}\tan\theta$$

by eliminating p. Substituting above G_2 turns at to be

$$G_2(q, P) = \frac{qP}{\cos\theta} - \frac{1}{2} m\omega\left(q^2 + \frac{P^2}{m^2\omega^2}\right)\tan\theta$$

Verify that

$$\left(\frac{\partial F_2}{\partial q}\right)_P = \frac{P}{\cos\theta} - m\omega q \tan\theta = p$$

$$\left(\frac{\partial F_2}{\partial P}\right)_q = \frac{q}{\cos\theta} - \frac{P}{m\omega}\tan\theta = Q$$

Ex. 3. We consider the harmonic oscillator where the transformed Hamiltonian K in terms of the variables Q and P can be enforced to be vanishing leading to the result that Q and P are constant in time. To this end we consider the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$$

Under the set of canonical transformations considered in example 2

we have already seen

$$G_2(q, P) = \frac{qP}{\cos\theta} - \frac{1}{2} m\omega\left(q^2 + \frac{P^2}{m^2\omega^2}\right)\tan\theta$$

Further $H(q, p)$ turns art to be invariant in that

$$H(q, p) = \frac{1}{2m} p^2 + \frac{1}{2} m\omega^2 q^2$$

$$\rightarrow \frac{1}{2m} P^2 + \frac{1}{2} m\omega^2 Q^2 = H(Q, P)$$

Also

$$\left(\frac{\partial G_2}{\partial t}\right)_{q,P} = \left[qP\sin\theta - \frac{1}{2} m\omega\left(q^2 + \frac{P^2}{m^2\omega^2}\right)\right]\sec^2\theta\,\dot{\theta}$$

$$= -\left(\frac{P^2}{2m\omega} + \frac{1}{2} m\omega Q^2\right)\dot{\theta}$$

$$= -H(Q, P)\frac{\dot{\theta}}{w}$$

Therefore

$$K(Q,P,t) = \left(1 - \frac{\dot{\theta}}{w}\right) H(Q,P)$$

which vanishes for $\dot{\theta} = w$ i.e. $\theta = wt$. As a consequence $\dot{Q} = \frac{\partial K}{\partial P} = 0$, $\dot{P} = -\frac{\partial K}{\partial Q} = 0$ so Q and P are constants namely

$$Q = Q_0, \quad P = P_0$$

Reverting to the old coordinates we get

$$q(t) = Q \cos \omega t + \frac{P_0}{m\omega} \sin \omega t$$

$$p(t) = -m\omega Q_0 \sin \omega t + P_0 \cos \omega t$$

where Q_0 and P_0 stand for the initial values of q and p respeicetively.

VIII. 3 Hamilton-Jacobi equation :

Taking cue from Ex. 3. just considered we look, more generally, for canonical tranformation, namely $(q, p) \to (\beta, \alpha)$, that restricts β and α to be constants in time. Alternatively, we are looking for a situation in which the transformed Hamiltonian emerges as a vanishing quantity. In this regard the Type 2 generating function $G_2(q\,;\alpha, t)$ proves suitable :

$$\beta = \frac{\partial G_2}{\partial \alpha}, \quad p = \frac{\partial G_2}{\partial q} \qquad \text{(VIII.25)}$$

It is clear that while the first equation in (VIII.25) gives q as a function of β, α and t, substituting it in the second equation of (VIII.25) gives p as a function of β, α and t :

$$q = g(\beta, \alpha\,; t) \quad p = h(\beta, \alpha : t) \qquad \text{(VIII.26)}$$

Further since we required β and α to be constants in time, K has to satisfy according to (VIII.1)

$$\frac{\partial K}{\partial \beta} = 0, \ \frac{\partial K}{\partial \alpha} = 0 \qquad \text{(VIII.27)}$$

with $\qquad K = H + \frac{\partial G_2}{\partial t}.$

(VIII.27) signals K to be at most a function of time. Choosing the generating function in such a way that K vanishes we find from (VIII.28)

$$H\left(q, \frac{\partial G_2}{\partial q}, t\right) + \frac{\partial G_2}{\partial t} = 0 \qquad \text{(VIII.28)}$$

Acutally G_2 is the new genreating function G_2^* for which $K = 0$ holds ; we have dropped the star without indulging in any loss of generality. (VIII.28) is the time-dependent Jacobi equation. It is a first-order differential equation involing the n coordinates q_i's and t. Associated with the $(n + 1)$ variables we expect $(n + 1)$ constants of motion namely $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}$. However we notice one curious thing in (VIII.28) that G_2 itself does not appear in it : only its partial derivatives do. So one of the constants has no bearing on the solution i.e. the solution has an additive constant. Disregarding such an irrelevant additive constant we write the complete integral of (VIII.28) in the form

$$S = S(q_1, q_2, \ldots q_n \,;\, \alpha_1, \alpha_2, \ldots \alpha_n \,;\, t) \qquad \text{(VIII.29)}$$

where it is ensured that none of the α's is of additive nature to the solution.

Let us clarify the above issue by considering the problem of a free particle. The Hamilton-Jacobi equation is obviously

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{\partial S}{\partial t} = 0$$

It can be solved to obtain

$$S(q, E, t) = \sqrt{2mE}\, q - Et$$

which is the complete integral and E is as non-additive constant playing the role of α. Further from (VIII.25)

$$\beta = \frac{\partial S}{\partial E} = C \quad \text{(say)}$$

i.e. $C = \sqrt{\frac{m}{2E}}q - t \quad \Rightarrow \quad q = \sqrt{\frac{2E}{m}}(t + C)$ and $p = \frac{\partial S}{\partial q} = \sqrt{2mE}$.

Next, let us consider the harmonic oscillator problem :

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right)$$

$p = \frac{\partial S}{\partial q}$ gives

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m^2\omega^2 q^2\right] + \frac{\partial S}{\partial t} = 0$$

We try for a solution adopting separation of variables :

$$S(q\,;\,\alpha\,;\,t) = W(q\,;\,\alpha) - \alpha t$$

Here α is a non-additive constant.

We get

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial q}\right)^2 + m^2\omega^2 q^2\right] = \alpha$$

which gives

$$W = \sqrt{2m\alpha}\int dq\sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}}$$

$\beta = \frac{\partial S}{\partial \alpha}$ gives

$$\beta = \sqrt{\frac{m}{2\alpha}}\int \frac{dq}{\sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}}} - t$$

Integrating

$$t + \beta = \frac{1}{\omega}\sin^{-1} q\sqrt{\frac{m\omega^2}{2\alpha}}$$

from which we get

$$q = \sqrt{\frac{2\alpha}{m\omega^2}}\sin\omega(t+\beta)$$

$$p = \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} = \sqrt{2m\alpha - m\omega^2 q^2} = \sqrt{2m\alpha}\,cos\,\omega(t+\beta))$$

A few remarks on the time-independent Hamilton-Jacobi equation. In this case we have from (VIII. 29)

$$H\left(q, \frac{\partial G_2}{\partial q}\right) + \frac{\partial G_2}{\partial t} = 0 \qquad \text{(VIII.30)}$$

We solve (VIII.30) by means of separation of variables by writing

$$G_2(q,t) = W(q) + T(t) \qquad \text{(VIII.31)}$$

Then (VIII.30) reduces to the combination

$$H\left(q, \frac{\partial W}{\partial q}\right) = E, \quad \frac{dT}{dt} = -E \qquad \text{(VIII.32)}$$

The first of (VIII.32) is called the time-independent Hamilton Jacobi equation. Note that the constant E has to belong to the set of n non-additive α's $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$. If this set does not include E then it must be a function oif the α's namely $E = E(\alpha_1, \alpha_2, \ldots \alpha_n)$. Note that E represents the total constant energy. We can then write

$$G_2 = W(q_1, q_2, \ldots, q_n;\ \alpha_1, \alpha_2, \ldots \alpha_n) - E(\alpha_1, \alpha_2, \ldots \alpha_n)t$$

with the canonical transformation

$$p = \frac{\partial W(q;\alpha)}{\partial q}, \ \beta = \frac{\partial W(q;\alpha)}{\partial \alpha} - \frac{\partial E(\alpha)}{\partial \alpha}t \qquad \text{(VIII.33)}$$

(VIII.33) may be intertreted as a transformation from (q, p) to (Q, P) where

$$Q = \beta + \frac{\partial E}{\partial \alpha}t, \ P = \alpha$$

Since $E = E(\alpha_1, \alpha_2, \ldots \alpha_n) = E(\alpha) = E(P)$, the transformed Hamiltonian is $E(P)$. In other words the new coordinates are cyclic.

Let us consider the motion of a particle which is subject to the influence of a potential $V(x)$. From (VIII.32)

$$\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 + V(x) = E$$

i.e. $W = \int \sqrt{2m[E - V(x)]}\, dx$

Taking E to be the new momentum α, (VIII.33) gives

$$p = \frac{\partial W}{\partial x} = \sqrt{2m(E - V(x))}$$

$$\beta + t = \frac{\partial W}{\partial E} = \sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{2m(E - V(x))}}$$

The integration is straightforward for $V(x) = \frac{1}{2}m\omega^2 x^2$.

Putting $x = \sqrt{\frac{2E}{m\omega^2}} \sin\phi$ we obtain

$$W = \frac{E}{\omega}(\phi + \sin\phi\cos\phi)$$

Moreover

$$p = \frac{\partial W}{\partial x} = \sqrt{2mE}\cos\phi$$

$$\beta + t = \frac{\partial W}{\partial E} = \frac{\phi}{\omega}$$

VIII.4 Action-angle variables :

We now consider a bounded motion in a finite-dimensional space. If the procedure of separation of variables is successfully implemented then W can be expanded in terms of components each one corresponding to the separated variable. In other words we can write.

$$W(q\,;\alpha) = \sum_{i=1}^{n} W_i(q_i, \alpha) \qquad \text{(VIII.34)}$$

where of course $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_n)$. Actuall we can identify each component of α with the separation constants. Note that we are dealing with a system possessing n degrees of freedom and that the components of α stand for the new momenta which are constants.

Now

$$p_i = \frac{\partial W}{\partial q_i}$$

$$= \frac{\partial W_i(q_i;\alpha)}{\partial q_i}, \quad i = 1, 2, \ldots n \qquad \text{(VIII.35)}$$

from (VIII.34). Integrating

$$W_i = \int p_i(q_i;\alpha)dq_i, \quad i = 1, 2, \ldots n$$

If we take the trajectory of the coordinate q_i around a closed path, with other coordinates remaining spectators, then the change of W_i is given by

$$\Delta W_i = \oint p_i\, dq_i\,. \qquad \text{(VIII.36)}$$

The integrand reflects an area in the (q_i, p_i) phase plane enclosed by the trajectory. Denoting $\Delta W_i = 2\pi I_i$ we can also express (VIII.36) as

$$I_i = \frac{1}{2\pi}\oint p_i \, dq_i \tag{VIII.37}$$

Clearly $I_i = I_i(\alpha) = I_i(\alpha_1, \alpha_2, \ldots \alpha_n)$. It is called the action variable for the i-th degree of freedom.

It is possible to express W in terms of q and $I_1, I_2, \ldots I_n$ by inverting α's in terms of the I's. In such a case Jacobi complete integral is given by

$$W(q\,;I) = \sum_{i=1}^{n} W_i\left(q_i; I_1, I_2, \ldots I_n\right) \tag{VIII.38}$$

The coordinate ϕ_i conjugate to I_i is called the action variable and given by

$$\phi_i = \frac{\partial W(q\,;I)}{\partial I_i} \tag{VIII.39}$$

The dimensioon of I_i is that of the 'action' and $W(q\,;I)$ is, in efect, the generating function of the canonical transformation $(q, p) \rightarrow (\phi, I)$.

Ex. The potential of a two-dimensional oscillator is given by

$$V(x, y) = \frac{1}{2} m\,\omega_x^2 + \frac{1}{2} m\,\omega_y^2\, y^2$$

The Hamilton-Jacobi equation reads from (VIII.32)

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right] + \frac{1}{2} m\omega_x^2 x^2 + \frac{1}{2} m\omega_y^2 y^2$$
$$= E = \alpha_x + \alpha_y \ \text{(say)}$$

Trying the solution $W = X(x) + Y(y)$ we can solve the above equation to get

$$W = \int \sqrt{2m\,\alpha_x - m^2\omega_x^2\, x^2}\; dx + \int \sqrt{2m\,\alpha_y - m^2\omega_y^2\, y^2\; dy}$$

Subsequently

$$p_x = \frac{\partial W}{\partial x} = \sqrt{2m\alpha_x - m^2\omega_x^2\, x^2},\ \ p_y = \frac{\partial W}{\partial y} = \sqrt{2m\alpha_y - m^2\omega_y^2\, y^2}$$

From (VIII.37) the action variables are

$$I_x = \frac{1}{2\pi}\oint \sqrt{2m\alpha_x - m^2\omega_x^2\, x^2}\; dx,\ \ I_y = \frac{1}{2\pi}\oint \sqrt{2m\alpha_y - m^2\omega_y^2 y^2}\, dy$$

Substituting $x = \sqrt{\frac{2m\alpha_x}{m^2\omega_x^2}} \sin\phi$, $y = \sqrt{\frac{2m\,\alpha_y}{m^2\omega_y^2}} \sin\theta$ we easily obtain

$$I_x = \frac{1}{4\pi}\frac{2m\alpha_x}{m\omega_x}\int_0^{2\pi}(1+\cos 2\phi)d\phi = \frac{\alpha_x}{\omega_x}$$

$$I_y = \frac{\alpha_y}{\omega_y}$$

$$\therefore \quad \alpha_x = \omega_x I_x, \; \alpha_y = \omega_y I_y$$

So we can express W in terms of I_x and I_y :

$$W = \int\sqrt{2m\,\omega_x I_x - m^2\omega_x^2 x^2}\,dx + \int\sqrt{2m\,\omega_y I_y - m^2\omega_y^2 y^2}\,dy$$

The angle variables are

$$\phi_x = \frac{\partial W}{\partial I_x} = \int\frac{dx}{\sqrt{\left(\frac{2I_x}{m\omega_x}\right) - x^2}}, \quad \phi_y = \frac{\partial W}{\partial I_y} = \int\frac{dy}{\sqrt{\left(\frac{2I_y}{m\omega_y}\right) - y^2}}$$

Putting $x = \sqrt{\frac{2I_x}{m\omega_x}} \sin\epsilon$ and $y = \sqrt{\frac{2I_y}{m\omega_y}} \sin\delta$ we get ϕ_x and ϕ_y.

Unit : 1 ❑ Special Theory of Relativity

1.1. INTERTIAL FRAME

In physical science we generally study the temporal and spatial behaviour of the physical systems, and consequently the laws are expressed in respect of space and time. Any such a description requires some frame of reference, as for an example, a cartesian corrdinate system in three dimensions. Astronomy is considered to be one of the oldest branch of science, in which the frame of reference plays the pivotal role. Before Copernicus (1475–1543) the earth was considered to be at the centre of the universe, and the heavenly bodies including the sun were negarded as to be moving around the earth. This geocentric reference frame was replaced by Copernicus and also by Galileo by the heliocentric frame of reference in which the sun received the central position (the origin of the coordinate system) with all the planets moving around it as the satellites. Later, Newton's description of the law of physics in regrards to this helioentric frame of reference gaves its profound foundation. In fact, Mechanics, one of the branch of physics required frame of reference most. Of these frames of references, the inertial frame is the important one in which we may get laws of nature in their simpler forms.

Before Newton, it was believed that the earth and the natural state of things on it were at rest, that is, any object remained in the state of rest as long as no outside force acted on it. In Newton's formulation both the state of rest and the uniform motion in a straight line got equal importance. In fact, Newton's first law states : a body is either at rest or in uniform motion in a striaght line if no external force acts on it. the inertial frame is the one in which this law of Newton holds good. In the absence of gravitational or other force fields, we can have an inertial frame in which if the particle is set in motion it will move with steady speed in a straight line. One the other hand, if the particle under no force does not remain at rest or in uniform motion in a straight, the farme is not inertial. As an example, the frame of refernece fixed in the stars is an inertial frame. But a ccordinate frame fixed in the earth is not an inertial frame because of the fact the easth is spining about its axis as will as in moving around the sun.

1.2. GALILEAN TRASFORMATION

Galilian transformation is the transformation which relates the two inertial frames. Let (x, y, z) be the coordinates in the first intertial frume and t represents the time measured by the clocks attached in this frame. We call the set (x, y, z, t) the unprimed set. For the second inertial frame the primed set (x´, y´, z´, t´) represents the corresponding coordinates and time. Let us suppose that these two sets of coordinates system are parallel to each other, that is, the x´ axis is parallel to x axis, the y' axis to the y axis and the z´ axis to the z axis. Then the following transformation

$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\} \quad (1.2.1)$$

is the Galilean transformtion. This represents the motion of the primed frame in the direction of x axis with respct to the unprimed frame. v is the velocity, and it is supposed that the origins of the two sets of coordinates coincide at $t = t' = 0$.

The more geneal Galilean transformation between two frames representing uniform translatory motion relative to each other is given by

$$\left.\begin{aligned} x' &= x - V_x t \\ y' &= y - V_y t \\ z' &= z - V_z t \\ t' &= t \end{aligned}\right\} \quad (1.2.2)$$

Here, V_x, V_y, and V_z are the components of the velocity of the primed frame relative to the unprimed frame, We have chosen here also that their corresponding axes are paralled to each other, and their origins coincide at $t = t' = 0$. The two inertial frames may be connected by the following transformation which is also a Galilean transformation :

$$\left.\begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z + \alpha \\ y' &= a_{21}x + a_{22}y + a_{23}z + \beta \\ z' &= a_{31}x + a_{32}y + a_{33}z + \gamma \\ t' &= t \end{aligned}\right\} \quad (1.2.3)$$

Here, (α, β, γ) are the coordinates of the origin of the unprimed frame with respect to the primed frame. The constants a_{ij} are the cosines of the angles between the axes of the unprimed and primed frames. Note that the first law of Newton, the law of inertial, is expressed in the following mathematical form :

$$\ddot{x} = 0, \quad \ddot{y} = 0, \quad \ddot{z} = 0 \qquad \ldots\ldots (1.2.4)$$

(the notation '$\cdot$' respresnets 'derivative' with respect to the time, ie.,

$$\dot{x} = \frac{dx}{dt}, \quad \ddot{x} = \frac{d^2x}{dt^2}\Big)$$

The first integral of (1.2.4) expresses the law of inertial in its usual form which is

$$\dot{x} = u_0, \quad \dot{y} = v_0 \quad \dot{z} = w_0 \qquad \ldots\ldots (1.2.5)$$

where the constants u_0, v_0, w_0 represnt the initial values of the components of velocities along the three axes. These constant may by zero. Thus, if initially the body is at rest then if remains always at rest under no external force. On the contrary, if the body is in motion initially its velocity remains the same always. Clearly, it moves in a straight line. Also it is apparent that the law of inertia (1.2.4) is covariant with respect to the Galilean transformations (1.2.2) and (1.2.3).

Now, form the Galilean transformation (1.2.2) we can have, on differentiation with respect to time

$$\left.\begin{aligned} \dot{x}' &= \dot{x} - V_x \\ \dot{y}' &= \dot{y} - V_y \\ \dot{z}' &= \dot{z} - V_z \end{aligned}\right\} \quad (1.2.6)$$

Thus, the velocity vector $\vec{v} = (\dot{x}, \dot{y}, \dot{z})$ and $\vec{v}' = (\dot{x}', \dot{y}', \dot{z}')$ of the two inertial frames are connected by

$$\vec{v}' = \vec{v} - \vec{V} \qquad \ldots\ldots (1.2.7)$$

where (V_x, V_y, V_z)

Again, by differentiation with respect to time (remembering that $t = t'$) we have from (1.2.7)

$$\frac{d\vec{v}'}{dt'} = \frac{d\vec{v}}{dt} \quad \text{or,} \quad \vec{a}' = \vec{a} \quad \ldots\ldots (1.2.8)$$

that is, the acceleration remains invariant under Galilean transformation. It is to be noted that the time measuned in the inertial frame has the same value, that is, the operation of clocks is independent of the speed of the inertial frames with respect to one another. But we shall see later that a distinction must arise in the measurements of time in any two inertial frames for the case of the special theory of relativity where the Lorentz transformation will replace the Gaiilean transformation.

Example 1.2.1. The water in a river moves west at the speed of 6 mph and a boat heads north at 8 mph with respect to the water. Find out the direction and velocity of the boat with respect to the ground.

Solution : We take west as the positive x direction (see fig 1.2.1)

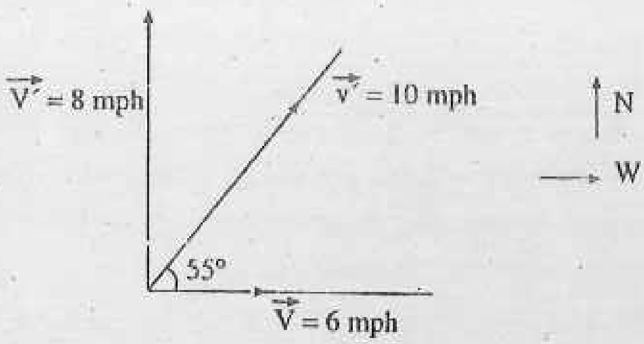


Fig. 1.2.1

and north as the postive y axis. Then the motion of the boat relative to the primed frame (fixed in the water) is given by

$$\bar{v}' = 8j \quad (\text{mph})$$

The motion of water with respect to the ground is given by

$$\bar{v}' = 6i \quad (\text{mph})$$

Here, i and j represent the unit vectors along the directions of x and y axes respectively. Then, motion of the boat with respect to the ground can be found by using (1.2.7). It is given as

$$\bar{v} = \bar{v}' + \vec{V}$$
$$= (6j + 8j)\ \text{mph}$$

Therefore, the speed with respect to the ground is $\sqrt{6^2 + 8^2} = 10$ mph, and the direction is $\tan^{-1}\frac{8}{6} = \tan^{-1}\frac{4}{3} = 53°$ north of west.

Exercise 1.2.1. After going a kilometer upstream in a motor boat, a man accidentally drops an oar overboard. He proceed upstream for 10 minutes after he missed the oar. He then turns round and retrieves the oar at the point from where he started initially. If the boat travels at constant speed with respect to the water, what is the speed of the current ? Work the problem twice, in the frame fixed in the bank, and in the frame fixed in the stream. **[Ans.** 3 Kmph]

1.3. VELOCITY OF LIGHT

As the speed of light is very large, the measurement of it requires either long paths or devices for measuring very short time intervals. Thus, an astronomical measurement of the velocity of light came first in 1675 by a Danish astronomer Roemer (1644 – 1710). He observed that a larger time elapsed between the eclipses of one of Jupiters moons during the period the earth is receding from Jupiter, the position A_1, in the figure 1.3.1, than during the period the earth is approaching Jupiter, position A_2.

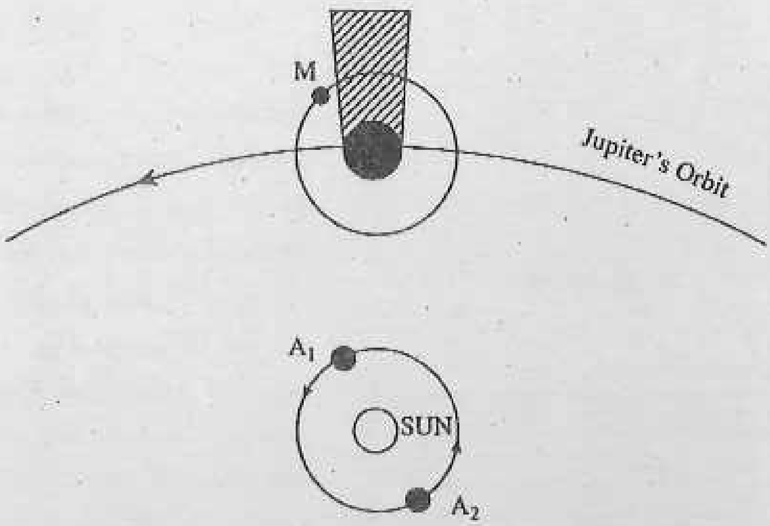


Fig. 1.3.1

This discrpancy was interpreted as due to the finite speed of light because light travelled larger distnce between eclipses the earth's receding period than that of the earth's approaching Jupiter. Roemer calculated the speed of light to be about $2{\cdot}3 \times 10^8$ m/sec. as light took 22 minutes to cross the earth's orbit, a distance of about 3×10^8 km. This value of the velocity of light c is close to the presently acepted value of it (within 25% of the present value 3×10^{10} cm/sec). Galileo also attempted the measurement of speed of light. Much later, Fizeau (1819–1896) modified the procedure of measurement employed by Galileo. At the time a preferred medium for light was assumed, and relative to this medium the velocity of light was taken as to be 'c'. This medium called as the "ether" filling the infinite space, which is perfectly transparent to light and nonresistant to the passages of all heavenly bodies. Now the question arises whether by a Galilean transformation one can get the velocity of light with respect to some other frame, say, with respect to the earth. In fact, the speed of the earth's orbital motion is about 3×10^6 cm/sec, and sensitive experiments might be performed to show that the speed of light with respect to the earth would depend on the direction of light travelled with respect to the earth's motion through the ether, in accordance to the Galilean transformation. Such an experiment was conducted by Michelson and Morley in 1887 in order to determine the motion of the earth through the ether, or alternatively, to test the applicability of the Galilean transformation to the motion of light.

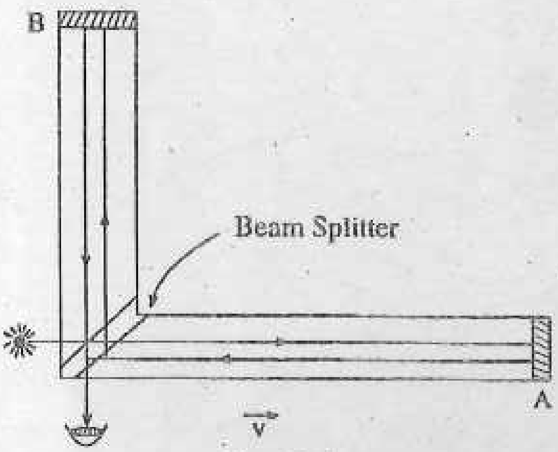


Fig. 1.3.2

In the experiment of Michelson and Morley, the parallel and perpendicular transit times were directly compared by using light waves as their own means of measuring time. Let us consider that the parallel and perpoendicular rods are set up as in Fig. 1.3.2. Then a beam of light from a source is split into two by a lightly silvered mirror, half the light moving down the rod B and the other half down rod A. On reflection the light returns down the rods and can be inspected by an observer looking through the same mirror at the direction of 90° to the source of light. As the light is a wave motion, the two beams start out in phase, or in step. If the beams return in step, then the time required for the two trips is the same. On the other hand, if they return out of step the time difference between the two trips is the time for half an oscillation. For the former case the observer notes a bright view, whereas for the latter case it is found to be dark.

It is to be noted that this set up of apparatus might have the necessary precision. For example, yellow light has a period of 2×10^{-15} sec., that is, the time for half an oscillation is about 10^{-15} sec. An interferometer in which the length of each section is about 15m yields an elapsed time of 10^{-7} sec. for the return trip. Therefore, a time measurement to half the period of oscillation of a light wave could measure the transit time to $10^{-15}/10^{-7}$, or to 1 part in 10^{8}, as required. In the actual experiment the path length employed was 11m.

Actually there was no independent way of determining that the length of the two paths was identical. The instrument was first adjusted to yield a bright view to the observer and then was rotated by 90° to detect any difference between the parallel and perpendicular transit times. Although the instrument was capable of detecting the time differences predicted by the Galilean transformation, no shift in the appearance of the field of view was detected that might be attributed to motion through the ether. This Michelson-Morley experiment was repeated many times in different laboratories using apparatus constructed of different materials. But each time the negative result was obtained. This failure of the experiment to find an ether questioned the validity of the Galilean transformation, and the special theory of relativity was proposed by Albert Einstein (1879–1959) in 1905 to resolve the issue. This theory asserts that no physical experiment can detect the absolute motion of an inertial reference frame.

1.4. SPECIAL THEORY OF RELATIVITY

The consequence of the results of the Micheloson-Morley experiment is that the Galilean transformation is not valid, and that velocities do not add vectorially at high speeds. Since no physical experiment can detect the absolute motion of an inertial reference frame there is no purpose in postulating the existence of an ether. Also, there is no purpose in speaking of a velocity of light except with respect to the observer who measures it. No observer is supposed to be a preferred one, that is, superior to all others. The velocity of light should have the same value to the observers of all inertial frames. The fundamental postulate of the theory of relativity is that "Physical law must have the same meaning in all inertial frames". This postulate is known as the postulate of covariance of physical law. Thus, the speed of light is the same in every inertial frame. Then the speed of light is the same in all direction and does not depend on the earth's motion in space.

1.5. LORENTZ TRANSFORMATION

We have seen that the Galilean transformtion is not valid at high speed but it is true in ordinary experience where one deals with low speeds. Therefore, in view of special theory of relativety a new transformation between two inertial frames is required. This transformation should satisfy the following conditions :

(i) The transformation must be linear ; that is, any single event in one inertial frame must transform to a single event in another frame, with a single set of coordinates.

(ii) In the limit of low speeds compared to the velocity c of light the transformation must approach the Galilean transformation.

(iii) The velocity of light must have the same value c in every inertial frame.

A flash of light (wave) spreads out as a growing sphere like circle ripples in the water of a pond.

The radius of this sphere grows at speed c, and the equation of the sphere is given by

$$x^2 + y^2 + z^2 = c^2t^2 \qquad \text{...... (1.5.1)}$$

Similarly, in the primed inertial frame it is also spreading sphere. If the flash of light takes place when $t = t' = 0$ and when the origins of the two frames coincide, then the equation of the sphere of light is given by

$$x'^2 + y'^2 + z'^2 = c^2t'^2 \qquad \text{...... (1.5.2)}$$

The coordiante transformtion which satisfies these requirements is known as Lorentz transformation.

If the axes of two inertial frames are parallel and the origins of these frames coincide at $t = t' = 0$, then Lorentz transformation between these frames is given by

$$\left.\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned}\right\} \quad (1.5.3)$$

where $$\gamma = \frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \qquad \ldots\ldots(1.5.4)$$

Here, it is taken as that the primed frame is moving with speed v in the positive direction with respect to the unprimed frame. The inverse transformation of (1.5.3), which is also a Lorentz transformation is

$$\left.\begin{aligned} x &= \gamma(x' + Vt') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + \frac{Vx'}{c^2}\right) \end{aligned}\right\} \quad (1.5.5)$$

In this case the unprimed frame is moving with speed v in the negative x' direction with respect to the primed frame. Clearly, the Lorentz transformations (1.5.3) and (1.5.5) satify above conditions. These are linear transforamtions, and for $v \ll c$, $\gamma \approx 1$, these transformations become the Galilean transformations. Thus, in the limit of low velocities the Lorentz transformation yields the results of ordinary experience.

Exercise 1.5.1. Show that the Lorentz transformation (1.5.3) satisfies the requirement (1.5.2) if (1.5.1) is taken into account.

Unit : 2 ❑ Simultaneity and Time Sequence

2.1. SIMULTANEITY AND TIME SEQUENCE

We have seen that the Lorentz transformation fulfills the condition (iii), that is, the velocity of light is the same in all inertial frames. In each frame a light sphere spreads out from the origin of the frame with speed c. Let us now examine some implications of the Lorentz transformations. Let the two events, 1 and 2, be simultaneous in the unprimed (inertial) frame. In order to ensure simultaneity of two events occurring at different places the obsevers must have synchronized identical clocks. Also, the observers should have access to identical meter sticks for measuring the position coordinates. In fact, each event is described by at least a set of four coordinates,the three position coordinates and the time. If the measurement of simultaneity of these two events is established, say at the time $t_1 = t_2$ where (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) are the four-coordinates of the 1st and 2nd events respectively, then from the Lorentz transformation (1.5.3) we have

$$t_1' = \gamma\left(t_1 - \frac{vx_1}{c^2}\right) \text{ and } t_2' = \gamma\left(t_2 - \frac{vx_2}{c^2}\right) \quad \ldots\ldots (2.1.1)$$

Therefore, $t_2' - t_1' = \gamma \frac{v}{c^2}(x_1 - x_2)$ ......(2.1.2)

From this relation we see that the two enents will be simultaneous, ie., $t_2' = t_1'$ if they occur at the same point, ie., at $x_1 = x_2$. The second event may be observed in the primed frame earlier or later than the first depending on their postions in the unprimed frame.

Let us suppose that $t_1 < t_2$, that is the first event preceeds the second in the unprimed frame. Then we have

$$t_2 - t_1 = \gamma\left[(t_2 - t_1) - \frac{v}{c_2}(x_2 - x_1)\right] \quad \ldots\ldots(2.1.3)$$

If R.H.S. of (2.1.3) is equal to zero, then $t_1' = t_2'$, that is, the two events are simultaneous in the primed frame. On the other hand, if it is less than zero, then the events are observed in reverse order in the two frame. This will occur if

$$x_2 - x_1 > \frac{c^2}{v}(t_2 - t_1) < c(t_2 - t_1) \quad \ldots\ldots (2.1.4)$$

This corresponds to the case of the two enents happening at such a distance apart that a ray of light leaving the first event could not have reached to the place of second event in time to cause that event.

Thus, we see that the time sequence of two events can only be reversed if they are not "causally connected", that is, one event can not cause the other by sending a signal at the speed c. As no signal can travel faster than light, therefore no signal can bridge the interval between the two events separated by the relation (2.1.4). Consequently, one event cannot have the knowledge of the prior occurrent of the other.

2.2 TIME DILATION

Let us suppose that a clock is at rest in the primed (inertial) frame. Such a frame is called the proper frame of the clock. This frame is supposed to be moving with speed v in the positive x-direction with respect to the unprimed frame, say, the laboratory. Therefore, the clock is also moving with speed v in the +x -direction. By application of the Lorentz transformation (15.5) we have

$$t_1 = \gamma\left(t_1' + \frac{vx_1'}{c^2}\right), \quad t_2 = \gamma\left(t_2' + \frac{vx_1'}{c^2}\right) \quad \text{......(2.2.1)}$$

where (x_1', y_1', z_1') is the fixed position of the clock in the primed frame, and t_1', t_2' are the two beats of the clock in that frame. The observed times in the unprimed frame corresponding to these beats are t_1 and t_2 respectively. Therefore, the time difference in the unprimed frame is given by

$$t_2 - t_1 = \gamma(t_2' - t_1') \quad \text{......(2.2.2)}$$

$$\text{Now, } \gamma = \frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} > 1 \quad \text{for} \quad 0 < v < c \quad \text{......(2.2.3)}$$

consequently, $t_2 - t_1 > t_2' - t_1'$

Thus, the time interval in the proper frame of clock will be smaller than the observed time interval in the laboratory frame. That is, the clock which is moving with respect to the laboratory runs slow. A proper time interval is dilated or expanded when measured from some inertial frame other than the proper frame itself.

Example 2.2.1. The subatomic particles, mesons, decay at an exponential rate such that $^1/_e$ of the original number remains after 2.6×10^{-8} sec. in a coordinate frame in which the mesons are at rest. Beams of π–mesons produced in an accelerator move at a speed of 99% of the velocity of light. Find the decay time of mesons in the laboratory frame. Calculate the average distance traversed by the meson beam before dropping to $^1/_e$ of its initial intensity.

Solution : $\frac{v}{c} = 0 \cdot 99, \quad \gamma = \frac{1}{\left\{1-(0 \cdot 99)^2\right\}^{\frac{1}{2}}} = 7 \cdot 18$

decay time of mesons in the laboratory

$$= 2{\cdot}6 \times 10^{-8} \times 7{\cdot}18 = 1{\cdot}87 \times 10^{-7} \text{ sec.}$$

meson beam moves with a speed $0{\cdot}99 \times 2{\cdot}998 \times 10^8$ m/sec

In decay time the beam moves a distance $1{\cdot}87 \times 10^{-7} \times 0{\cdot}99 \times 2{\cdot}998 \times 10^8$ m

$$= 56 \text{ m}$$

Exercise 2.2.1. A particle with a mean proper lifetime of 10^{-6} sec. moves through the laboratory at $2{\cdot}7 \times 10^{10}$ cm/sec. (a) What is its lifetime, as measured by observers in the laboratory ? (b) Calculate the average distance traversed by the particle in the laboratory before disintegrating. (c) Repeat the calculation of the preceding part without taking relativity into account.

[Ans. (a) $2{\cdot}3 \times 10^{-6}$ sec. (b) $6{\cdot}2 \times 10^4$ cm (c) $2{\cdot}7 \times 10^4$ cm]

Exercise 2.2.2. By taking differentials of the Lorentz transformation equations show that the quantity ds transforms to ds′ where $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ and $ds'^2 = c^2dt'^2 - dx'^2 - dy'^2 - dz'^2$.

2.3. LORENTZ CONTRATION

Let us find the length of a rod in the laboratory (unprimed) frame by placing it at rest along the x-axis. The difference in the x-coordinates of the ends of the rod gives the length l of it. Now, with respect to an inertial frame (primed) moving along negative x-axis with speed v, the rod will appear to move along positive x′-axis with speec' v. If $(x_1, 0, 0)$ and $(x_2, 0, 0)$ are the end points of the rod in the laboratory, then by Lorentz transformation we have

$$x_2 = \gamma(x_2' - vt'), \quad x_1 = \gamma(x_1' - vt') \qquad \text{......(2.3.1)}$$

where the end points $(x_1', 0, 0)$ and $(x_2', 0, 0)$ are measured by synchronized clocks at time t' in the primed frame, If l' is the length of rod in this frame, we have

$$l = x_2 - x_1 = \gamma(x_2' - x_1') = \gamma l' \qquad \text{......(2.3.2)}$$

The unprimed frame (laboratory) is the proper frame for the rod, and since $\gamma > 1$ we see that

$$l' < l = \text{proper length} \qquad \text{...... (2.3.3)}$$

Thus, the length of the moving rod (as appearing inthe primed frame) is less than the proper length. This effect is called Lonentz contraction.

Exercise 2.3.1. A rod has a length of 80 cm. When it is moving with a speed of 0·75 c along the direction of its length, find the length of the rod with respect to an observer at rest (Ans. 53 cm)

2.4 VELOCITY TRANSFORMATIONS

In order to find the Lorentz transformations of the velocities we take differentials of (1.5.3) and (1.5.5) to get

$$\left.\begin{aligned} dx' &= \gamma(dx - v\,dt) \\ dy' &= dy \\ dz' &= dz \\ dt' &= \gamma\left(dt - v\frac{dx}{c^2}\right) \end{aligned}\right\} \qquad \text{...... (2.4.1)}$$

and

$$\left.\begin{aligned} dx &= \gamma(dx' + v\,dt' \\ dy &= dy' \\ dz &= dz' \\ dt &= \gamma\left(dt' + \frac{v\,dx'}{c^2}\right) \end{aligned}\right\} \qquad \text{......(2.4.2)}$$

Now $u_x = \frac{dx}{dt}$ and $u'_x = \frac{dx'}{dt'}$ are the x component and x′-component of velocity of the particle in the unprimed and primed frames respectively. Similarly, u_y, u_z, and u'_y, u'_z are the other components of the velocity in the respective frames. Then, it is easy to find the following Lorentz transformation for the velocities :

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}, u'_y = \frac{u_y}{\left(1 - \frac{u_x v}{c^2}\right)}, u'_z = \frac{u_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} \quad ......(2.4.3)$$

and

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}}, u_y \frac{u'_y}{\left(1 + \frac{u'_x v}{c^2}\right)}, u_z = \frac{u'_z}{\gamma\left(1 + \frac{u'_x v}{c^2}\right)} \quad ......(2.4.4)$$

It is to be noted that the primed frame is moving in the +x direction with respect to the unprimed frame with velocity v. Also, in the nonrelativistic limit v << c or, $\beta = \frac{v}{c}$ << 1, the above equation reduce to the Galilean velocity transformation (1.2.7).

Now if the moving particle is a photon which is moving in the primed frame with a velocity c then its velocity in the unprimed frame is

$$u_x = \frac{c + v}{1 + \frac{cv}{c^2}} = c \quad ...... (2.4.5)$$

for any value of v, even if v = c. Thus, we see that the velocity of light is the same in all inertial frames.

Exercies 2.4.1. Two particle come toward each other, each with speed 0·9c with respect to the laboratory. What is their relative speed ? (Ans. 0·995c)

2.5. TRANSFORMATION OF ACCELERATION

Taking differentials of the velocity transformation equations (2.4.4) we have

$$\left.\begin{aligned}
du_x &= \frac{du'_x}{1+\frac{vu'_x}{c^2}} - \frac{v+u'_x}{\left(1+\frac{vu'_x}{c^2}\right)^2}\frac{v}{c^2}du'_x \\
du_y &= \frac{du'_y}{\gamma\left(1+\frac{vu'_x}{c^2}\right)} - \frac{u'_y}{\gamma\left(1+\frac{vu'_x}{c^2}\right)^2}\frac{v}{c^2}du'_x \\
du_z &= \frac{du'_z}{\gamma\left(1+\frac{vu'_x}{c^2}\right)} - \frac{u'_z}{\gamma\left(1+\frac{vu'_x}{c^2}\right)^2}\frac{v}{c^2}du'_x
\end{aligned}\right\} \quad \text{......} (2.5.1)$$

We shall consider the case where the particle is instantaneously at rest in the proper frame, that is

$$u'_x = u'_y = u'_z = 0 \qquad \text{......}(2.5.2)$$

But the instantaneous acceleration is not necessarily zero in the proper frame. In this case we have from (2.5.1)

$$\left.\begin{aligned}
du_x &= du'_x\left(1-\frac{v^2}{c^2}\right) = \gamma^{-2}du'_x \\
du_y &= \gamma^{-1}du'_y, \quad du_z = \gamma^{-1}du'_z
\end{aligned}\right\} \qquad \text{......}(2.5.3)$$

Now, from Lorentz transformation we have

$$t = \gamma\left(t' - \frac{vx'}{c}\right)$$

By taking differential we get

$$dt = \gamma dt'\left(1 - \frac{v}{c^2}\frac{dx'}{dt'}\right) = \gamma dt'\left(1 - \frac{v}{c^2}u'_x\right)$$

$$= \gamma dt' \text{ (since } u'_x = 0 \text{ in the present case)} \text{ ......}(2.5.4)$$

Dividing each of equations (2.5.3) by (2.5.4) we get the following transformation equations for the acceleration as

$$
\left.\begin{aligned}
a_x &= \frac{du_x}{dt} = \frac{1}{\gamma^3}\frac{du'_x}{dt'} = \frac{1}{\gamma^3}a'_x \\
a_y &= \frac{dy_y}{dt} = \frac{1}{\gamma^2}\frac{du'_y}{dt'} = \frac{1}{\gamma^2}a'_y \\
a_z &= \frac{du_z}{dt} = \frac{1}{\gamma^2}\frac{du'_z}{dt'} = \frac{1}{\gamma^2}a'_z
\end{aligned}\right\} \quad \ldots\ldots(2.5.5)
$$

The above equations can be written without specific reference to an axis system. We may replace the subscript x by one which indicates that we are considering a component parallel to the motion of the particle as observed from the laboratory frame. Similary, the subscripts y, z are replaced by a subscript indicating the component perpendicular to the motion. One can then write

$$
\left.\begin{aligned}
a_{11} &= \gamma^{-3}a'_{11} \\
a_{\perp} &= \gamma^{-2}a'_{T}
\end{aligned}\right\} \quad \ldots\ldots(2.5.6)
$$

2.6. MOMENTUM AND FORCE

In the nonrelativistic case the momentum of a particle of mass m moving with the velocity $\bar{v}$ is given as

$$\bar{p} = m\bar{v} \quad \ldots\ldots(2.6.1)$$

In the relativity theory the momentum of a particle in an inertial frame should be defined in such a way that in the nonrelativistic region $|\bar{v}| << c$, $\beta \to 0$ or $c \to \infty$, the momentum must be given as in (2.6.1). It is, in fact, given by

$$\bar{p} = m\gamma\bar{v} \quad \ldots\ldots(2.6.2)$$

Clearly, $$\gamma = \frac{1}{\left(1 - \frac{\bar{v}^2}{c^2}\right)^{\frac{1}{2}}} \to 1 \text{ as } c \to \infty$$

Thus, the relation (2.6.2) corresponds to (2.6.1) of the nonrelativistic case.

Now, we can interprete γm as the mass of the particle which is moving with the velocity $\bar{v}$ in an inertial frame. Then the usual definition (2.6.1) holds good for the relativistic case. In an inertial frame which is moving with velocity $\bar{v}$ with respect to the former one the particle must be at rest, and consequently $\gamma = 1$. Thus, m is the mass of the particle at rest in an inertial frame (the later one). This frame is called the rest frame of the particle, and m is its rest mass. The mass of the particle which is moving with velocity $\vec{v}$ is given as

$$m\gamma = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}, \quad v = |\vec{v}| \qquad ......(2.6.3)$$

The momentum $\bar{p}$ and force $\vec{F}$ are related by Newton's second law. In fact, we know that

$$\vec{F} = \frac{d\bar{p}}{dt} \cdot \qquad ......(2.6.4)$$

In the nonrelativisitic limit the momentum is given by (2.6.1).

Therefore,

$$\vec{F} = \frac{d}{dt}(m\bar{v}) = m\frac{d\bar{v}}{dt} \text{ (for constant mass m)}$$

$$= m\bar{a}$$

But for the relativity theory we have to use the relation (2.6.2) for momentum and consequertly one has

$$\vec{F} = \frac{d}{dt}\left(m\gamma\bar{v}\right) = m\gamma\frac{d\bar{v}}{dt} + m\bar{v}\frac{d\gamma}{dt} \qquad ......(2.6.5)$$

The second term of R.H.S. in (2.6.5) is in the direction of the velocity vector. The first term is the derivative of the velocity vector with respect to time. This deritative may be in any direction. So we resolve it into a component perpendicular to the velocity and a component parallel to the velocity. Writing the perpendicular component as the perpendicular component of the acceleration, that is, writing

$$\left(\frac{d\bar{v}}{dt}\right)_{\perp} = a_{\perp}$$

we find $F_{\perp} = m\gamma a_{\perp}$ ......(2.6.6.)

To find the component of the force parallel to the velocity we first compute $\frac{d\gamma}{dt}$.

We have $\frac{d\gamma}{dt} = \frac{d}{dt}\left\{\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right\}$

or, $\frac{d\gamma}{dt} = -\frac{1}{2}\left(-\frac{1}{c^2}\right)\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\frac{d(v^2)}{dt} = \frac{1}{2c^2}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\frac{d(v^2)}{dt}$

Now, $v^2 = \vec{v}^2$

Therefore, $\frac{d(v^2)}{dt} = \frac{d\left(\vec{v}^2\right)}{dt} = 2\vec{v}.\frac{d\vec{v}}{dt} = 2\vec{v}.\vec{a} = 2v|\vec{a}|\cos\theta$

where θ is the angle between $\vec{v}$ and $\vec{a}$. Thus, $|\vec{a}|\cos\theta$ is the component of the acceleration in the direction of the velocity, that is a_{11}. Then $\frac{d\left(v^2\right)}{dt} = 2va_{11}$

Hence $\frac{d\gamma}{dt} = \frac{1}{2c^2}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}2va_{11} = \frac{v}{c^2}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}a_{11}$

Consequently, $F_{11} = mv\frac{v}{c^2}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}a_{11} + m\gamma a_{11}$

$$= a_{11}\left\{\frac{mv^2}{c^2}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} + \frac{m}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}\right\}$$

$$= a_{11}\frac{m}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left\{\frac{v^2}{c^2} + 1 - \frac{v^2}{c^2}\right\}$$

$$= a_{11}\frac{m}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} = m\gamma^3 a_{11} \quad \ldots\ldots(2.6.7)$$

Equations (2.6.6) and (2.6.7) give respectively the components of force, perpendicular and parallel to the direction of motion. The mass $m\gamma^3$ is called longitudinal mass, whereas $m\gamma$ is the transverse mass. The difference in these two masses implies that at high speed (that is, in the relativistic region) the acceleration vector is not parallel to the force vector.

2.7. ENERGY OF A PARTICLE

The work, dW, done by a force $\vec{F}$ moved through a small displacement $d\vec{s}$ is given by

$$dW = \vec{F}.d\vec{s} \qquad \text{......(2.7.1)}$$

We note that $d\vec{s} = \vec{v}\,dt$

and $\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\gamma\vec{v})$ ......(2.7.2)

Therefore $dW = \frac{d}{dt}\left(m\gamma\vec{v}\right).\vec{v}\,dt$ ......(2.7.3)

Then the work done in replacing the particle from rest "at position s = 0" to a final velocity $\vec{v}$ "at position s" is given by

$$W = \int_0^s \vec{F}.d\vec{s} = \int_0^s \frac{d}{dt}\left(m\gamma\,\vec{v}\right).\vec{v}\,dt$$

$$= \int_0^v d(m\gamma\vec{v}).\vec{v} = \int_0^v \left\{m\gamma(d\vec{v}).\vec{v} + (d\gamma)m\vec{v}.\vec{v}\right\}$$

Now, $d\gamma = d\left\{\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}\right\} = \frac{1}{c^2}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\vec{v}.d\vec{v}$

Therefore, $W = \int_0^v \left\{m\gamma\vec{v}.d\vec{v} + \frac{1}{c^2}\frac{mv^2}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\vec{v}.d\vec{v}\right\}$

$$= m\int_0^v \left\{ \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} + \frac{\frac{v^2}{c^2}}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} \right\} \bar{v}.\, d\bar{v}$$

$$= \frac{mc^2}{2}\int_0^v \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\, d\left(\frac{v^2}{c^2}\right) \quad \left[\text{since } \bar{v}.\, d\bar{v} = \frac{1}{2}d\bar{v}^2 = \frac{1}{2}dv^2\right]$$

On integration, we find

$$W = \left. \frac{mc^2}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} \right|_0^v = \frac{mc^2}{\left(1-\frac{v^2}{c^2}\right)} - mc^2$$

or, $$W = m\gamma c^2 - mc^2 \qquad \text{......(2.7.4)}$$

The second term on R.H.S. of (2.7.4) contains no velocity and we call it the rest energy or mass energy of the paraticle. We say that the particle of mass m has the energy E when it is moving with velocity $\bar{v}$ and this energy is given as

$$E = m\gamma c^2 \qquad \text{......(2.7.5)}$$

The kinetic energy T is the energy acquired by the particle as a result of the work done on it in raising its speed from 0 to v. This kinetic energy is the difference between the energy E and the rest energy mc^2. Thus, the kinetic energy T of a particle of mass m moving with velocity $\bar{v}$ is given by

$$T = E - mc^2 = mc^2(\gamma - 1) \qquad \text{......(2.7.6)}$$

Now, we can find the relation between energy and momentum.

Since, $E = m\gamma c^2$ and $\bar{p} = m\gamma\bar{v}$

we have $$E^2 - \bar{p}^2c^2 = m^2\gamma^2c^4 - m^2\gamma^2c^2\,\bar{v}^2$$

$$= m^2c^2\{\gamma^2c^2 - \gamma^2v^2\}$$

$$= m^2\gamma^2c^4\left(1 - \frac{v^2}{c^2}\right)$$

$$= m^2c^4$$

or, $$E^2 = \vec{p}^2c^2 + m^2c^4 \qquad \text{...... (2.7.7)}$$

Example 2.7.1. In the limit of low velocities show that the usual expression for kinetic energy can be obtained from the relativistic expression of it.

Solution : For v << c, we have

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \simeq 1 + \frac{1}{2}\frac{v^2}{c^2}$$

$$T = mc^2(\gamma - 1) = mc^2\frac{1}{2}\frac{v^2}{c^2} = \frac{1}{2}mv^2$$

Unit : 3 ❑ Elastic Solid Media

3.1. DEFORMATION OF A CONTINUUM :

Any material body is composed of a large number of discrete molecules separated from one another by empty space comparable with the molecular size. The average spacing of molecules differs in the physical states of matter, that is, in solid, liquid and gas. In solids these molecules are more closely packed than those in liquids and gases. Although these materials are essentially discontinuous and consist of sufficiently large number of discreate molecules, the distance between two neighbouring molecules are very small compared to the dimension of the body. Consequently, the average behaviour of the whole body made up of such large number of molecules is important, and we can disregard the actual discrete structure of it. We, thus, can have a continuum structure or a "continuum" which is a continuously distributed matter completely filled up in a negion of space.

A material point of a "continuum" or a particle is the matter contained in an infinitesimally small volume whose physical dimensions are so small that one can regard it to be a spatial point. Thus, any material point can be associated to a spatial point. The physical properties like density, displacement, velocity etc. are uniquely assigned for these material points of the continuum. Obviously, these physical properties are expressed as continuous functions of position and, of course, of time.

Let us consider the configuration the body, which is the complete specification of the positions of all material points of it at a given time. This specification is the region of space of certain volume having a boundary surface. This region contains the continuous body at a given time. Let us suppose that at $t = 0$ the region of space B_0 having its surface S_0 and volume V_0 is the configuration of the continuum. Due to the application of external forces this configuration may change to (B, S, V) (say) at a subsequent time t. As the consequence the positions of all material points of the body are changed. This change of configuration of the body is called "deformation".

3.2. LAGRANGIAN AND EULERIAN METHODS OF DESCRIBING DEFORMATIONS

In Lagrangian or material method of deformation is described by "following" the motion of material points. Each material point of the continuum is identified by the rectangular cartesian coordinates (X_1, X_2, X_3) of its position in its initial undeformed state. In this desription the physical properties are assigned to these material points labelled by these co-ordinates at the initial configuration, Consequently, all these physical properties must be function of these co-ordinates and the time t. The position of the material point in the deformed state at the time t, given by the rectangular cartesian co-ordinates (x_1, x_2, x_3), must also be functions of X_1, X_2, X_3 and t, that is,

$$x_i = x_i (X_1, X_2, X_3, t) \quad i = 1, 2, 3 \qquad \text{......(2.1)}$$

The co-ordinates (X_1, X_2, X_3) which are independent co-ordinates are called Lagrangian (or material) co-ordinates. On the other hand, the dependent co-ordinates (x_1, x_2, x_3) are called spatial co-ordinates.

In the Eulerian or spatial method the fixed spatial points replace the moving material points of Lagrangian method in describing the deformation. Each spatial point in the Eulerian method is occupied by different material points at different times, and one can observe the changes of various properties at the spatial point. In other words, the spatial points are endowed with physical properties, and the material points occupying a spatial point in different times acquire these physical properties of that point. Thus, the physical properties are functions of the rectangular cartesian co-ordinates (x_1, x_2, x_3) of the spatial point and of the time t. In paritcular, we have

$$X_i = X_i (x_1, x_2, x_3, t) \qquad i = 1, 2, 3 \qquad \text{......(2.2)}$$

and $$v_i = v_i (x_1, x_2, x_3, t) \qquad i = 1, 2, 3 \qquad \text{......(2.3)}$$

where the material point which was at the position (X_1, X_2, X_3) in the undeformed state at $t = 0$, passes the spatial position (x_1, x_2, x_3) with the velocity (v_1, v_2, v_3) at the time t (in the deformed state). In is to be noted that in Eulerian method the initial co-ordinates (X_1, X_2, X_3) are irrelevant, and the spatial or Eulerian co-ordinates (x_1, x_2, x_3) where the material points resides at time t serve the purpose.

For a fixed time t we have from equations (2.1) and (2.2)

$$\left.\begin{aligned} x_i &= x_i \ (X_1, X_2, X_3) \\ \text{and} \quad X_i &= X_i \ (x_1, x_2, x_3), \quad i = 1, 2, 3 \end{aligned}\right\} \quad \ldots\ldots(2.4)$$

These equations relate initial configuration to a subsequent configuratoin, and thus, the deformation is characterized by these equations. If in a deformation there are no changes in the relative positions of constituent material points of the continuum body, that is, the length of any line joining any two material points remains unchanged, then the deformation is a combination of translation and a rotation about an axis at any point causing no change in shape of the body. Such a defomation is known as rigid-body deformation, and the body is called a rigid body. On the contrary, the deformation which causes a change in the shape of the body, that is, the relative positions of material points of it change, is called a "strain deformation", and the body is said to be "deformable". Such deformation is possible in the case of occurrence of relative displacement of points in the continuum with respect to each other, that is in the case of changes of the length and orientation of any line joining any two ponits.

The displacement u_i of the material point (X_1, X_2, X_3) from its initial $(t = 0)$ undeformed state to the position (x_1, x_2, x_3) in the deformed state at time t is defined by

$$u_i = x_i - X_i, \quad i = 1, 2, 3 \qquad \ldots\ldots(2{,}5)$$

In should be noted that in the Lagrangian method u_i and x_i are functions of X_1, X_2, X_3 and time t. Therefore

$$u_i \ (X_1, X_2, X_3, t) = x_i \ (X_1, X_2, X_3, t) - X_i \quad \ldots\ldots(2.6)$$

On the other hand, in the Eulerian description , u_i and X_i are functions of x_1, x_2, x_3 and t. Thus,

$$u_i(x_1, x_2, x_3, t) = x_i - X_i \ (x_1, x_2, x_3, t) \qquad \ldots\ldots(2.7)$$

3.3. MEASURES OF FINITE STRAIN DEFORMATION IN LAGRANGIAN METHOD :

We consider the finite strain deformation from the initial undeformed configuration (B_0, S_0, V_0) to the deformed configuration (B, S, V). Let P_0 and Q_0 be two neighbouring material points (see fig. 1) in the undeformed configuration, and dL be the length of the line element P_0Q_0 which is oriented in the direction (N_1, N_2, N_3). The co-ordinates of P_0 and Q_0 are given respectively as (X_1, X_2, X_3) and $(X_1 + dX_1, X_2 + dX_2, X_3 + dX_3)$ with respect to an orthogonal set of co-ordinate axes fixed in space. In the deformed state the positions of these two points will be P and Q in the region B at time t. Let dl be the length of the line element PQ oriented in the direction (n_1, n_2, n_3), and the co-ordinates of P and Q are (x_1, x_2, x_3) and $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$ respectively with respect to the same set of co-ordinate axes.

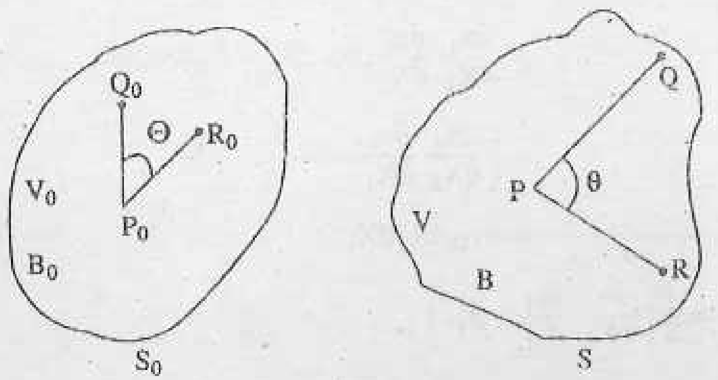


Fig. 1

Therefore, we have

$$dL^2 = \sum_{i=1}^{3}(dX_i)^2 = dX_i dX_i \qquad \text{(using summation convention)}$$

$$= \delta_{ij}\, dX_i\, dX_j \quad \dots (3.1)$$

where δ_{ji} is the Kronecker delta defined as

$$\left.\begin{aligned}\delta_{ij} &= 1,\ i = j\\ &= 0,\ i \neq j\end{aligned}\right\} \quad \dots (3.2)$$

Also, we have $N_i = \frac{dX_i}{dL}$, $i = 1, 2, 3 \quad \dots (3)$

Again, $dl^2 = \delta_{ij}\, dx_i\, dx_j \quad \dots (3.4)$

and $n_i = \frac{dx_i}{dl}$, $i = 1, 2, 3 \quad \dots (3.5)$

Now, (2, 1) characterizes the deformation, we have form this equation,

$$dx_i = \frac{\partial x_i}{\partial X_j} dX_j \quad \dots (3.6)$$

Substituting this relation into (3.4) we have

$$dl^2 = \delta_{ij} \frac{\partial x_i}{\partial X_k} dX_k \frac{\partial x_j}{\partial X_l} dX_l = \frac{\partial x_i}{\partial X_k} \frac{\partial x_i}{\partial X_l} dX_k dX_l \quad \dots\dots (3.7)$$

Consequently, $dl^2 - dL^2 = \frac{\partial x_i}{\partial X_k} \frac{\partial x_i}{\partial X_l} dX_k dX_l - \delta_{ij} dX_i dX_j$

$$= \frac{\partial x_i}{\partial X_k} \frac{\partial x_i}{\partial X_l} dX_k dX_l - \delta_{kl} dX_k dX_l$$

$$= \left(\frac{\partial x_i}{\partial X_k} \frac{\partial x_i}{\partial X_l} - \delta_{kl} \right) dX_k dX_l$$

$$= 2 r_{kl} dX_k dX_l \quad \dots\dots (3.8)$$

where $r_{kl} = \frac{1}{2}\left(\frac{\partial x_i}{\partial X_k} \frac{\partial x_i}{\partial x_l} - \delta_{kl} \right) \quad \dots\dots (3.9)$

We can also write

$$\frac{dl^2 - dL^2}{dL^2} = 2 r_{kl} \frac{dX_k}{dL} \frac{dX_l}{dL} = 2 r_{kl} N_k N_l \quad \dots\dots (3.10)$$

If u_i be the displacement of the material point from its position P_0 to P, then

$$u_i = x_i - X_i \quad \dots\dots (3.11)$$

Similarly, the dispacement $u_i + du_i$ of the material point Q_0 to the point Q is given by

$$u_i + du_i = (x_i + dx_i) - (X_i + dX_i) \quad \dots\dots, (3.12)$$

Therefore, using (3.11),

$$x_i - X_i + du_i = x_i - X_i + dx_i - dX_i$$

or, $dx_i = du_i + dX_i \quad \dots\dots (3.13)$

$$\frac{\partial x_i}{\partial X_k} = \frac{\partial u_i}{\partial X_k} + \delta_{ik} \quad \dots\dots (3.13a)$$

Therefore, from (3.9) it follows that

$$r_{kl} = \frac{1}{2}\left\{\left(\frac{\partial u_i}{\partial X_k} + \delta_{ik}\right)\left(\frac{\partial u_i}{\partial X_l} + \delta_{il}\right) - \delta_{kl}\right\}$$

$$= \frac{1}{2}\left\{\frac{\partial u_l}{\partial X_k} + \frac{\partial u_k}{\partial X_l} + \frac{\partial u_i}{\partial X_k}\frac{\partial u_i}{\partial X_l}\right\} \quad \ldots\ldots (3.14)$$

Again, the change in the angle between two line elements due to the deformation can be calculated. Let Θ be the angle between the line elements P_0Q_0 and P_0R_0 in the undeformed state. In the deformed state these line elments are PQ and PR respectively and θ is the angle between them. Let the lengths of P_0R_0 and PR be δL and δl respectively, and let the co-ordinates of R_0 and R be respectively $(X_1 + \delta X_1, X_2 + \delta X_2, X_3 + \delta X_3)$ and $(x_1 + \delta x_1, x_2 + \delta x_2, x_3 + \delta x_3)$.

Then $\cos\Theta = \dfrac{dX_i}{dL}\dfrac{\delta X_i}{\delta L}$ ...... (3.15)

Note that $M_i = \dfrac{\delta X_i}{\delta L}$ $(i = 1, 2, 3)$ represent the direction P_0R_0. The line PR is oriented in the direction (m_1, m_2, m_3) where $m_i = \dfrac{\delta x_i}{\delta l}$.

Then $\cos\theta = \dfrac{dx_i}{dl}\dfrac{\delta x_i}{\delta l}$ ...... (3.16)

Now, $\delta x_k = \dfrac{\partial x_k}{\partial X_j}\delta X_j$

Therefore, $\dfrac{\delta l^2 - \delta L^2}{\delta L^2} = 2r_{ij}\dfrac{\delta X_i}{\delta L}\dfrac{\delta X_j}{\delta L} = 2r_{ij}M_iM_j$

and $\dfrac{dl^2 - dL^2}{dL^2} = 2r_{ij}N_iN_j$ ...... (3.17)

Also, $dx_i\delta x_i - dX_i\delta X_i = dx_k\delta x_k - dX_i\delta X_i$

$$= \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j}dX_i\delta X_j - dx_i\delta X_i$$

$$= \left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right)dX_i\delta X_j$$

$$= 2r_{ij}dX_i\delta X_j$$

or, $\frac{dx_i}{dL}\cdot\frac{\delta x_i}{\delta L}-\frac{dX_i}{dL}\frac{\delta X_i}{\delta L}=2r_{ij}\frac{dX_i}{dL}\frac{\delta X_j}{\delta L}$

or, $\frac{dx_i}{dl}\cdot\frac{\delta x_i}{\delta l}\cdot\frac{dl}{dL}\frac{\delta l}{\delta L}-\frac{dX_i}{dL}\frac{\delta X_i}{\delta L}=2r_{ij}\frac{dX_i}{dL}\frac{\delta X_j}{\delta L}$

or, by using (3.15) and (3.16),

$$\frac{dl}{dL}\frac{\delta l}{\delta L}\cos\theta-\cos\Theta=2r_{ij}N_iM_j \quad \text{......} \ (3.18)$$

When $r_{ij}=0$, we see from (3.10) and (3.17) that $dl = dL$ and $\delta l = \delta L$. Consequently, from (3.18) we have $\theta = \Theta$.

Thus, the length of the line element and the angle between two line elements remain unchanged during the deformation, that is, it is a rigid body deformation. The necessary and sufficient condition for a rigid body deformation at each point is, thus, $r_{ij} = 0$. A nonzero tensor r_{ij} represents strain deformation and r_{ij} is known as strain tensor, a finite strain tensor. It is apparent that r_{ij} is symmetric, i.e., $r_{ij} = r_{ji}$.

3.1. MEASURES OF FINITE STRAIN DEFORMATION IN EULERIAN METHOD

We, now, consider deformation in Eulerian method in which the spatial co-ordinates (x_1, x_2, x_3) and time t are regarded as independent variables. Also the deformation is given by

$$X_k = X_k\ (x_1, x_2, x_3, t)$$

Therefore, $dX_k=\frac{\partial X_k}{\partial x_j}dx_j$ ...... (4.1)

$$dL^2 = dX_k\,dX_k=\frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}dx_i dx_j$$

and $dl^2 = \delta_{ij}\,dx_i\,dx_j$

Consequently, $\frac{dl^2-dL^2}{dl^2}=2\eta_{ij}n_in_j$ ...... (4.2)

where $\eta_{ij}=\frac{1}{2}\left(\delta_{ij}-\frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right)$ ...... (4.3)

Since $dX_k = dx_k - du_k$, we have

$$\frac{\partial X_k}{\partial x_i} = \delta_{ki} - \frac{\partial u_k}{\partial x_i}, \text{ and therefore}$$

$$= \eta_{ij} = \frac{1}{2}\left\{\delta_{ij} - \left(\delta_{ki} - \frac{\partial u_k}{\partial x_i}\right)\left(\delta_{kj} - \frac{\partial u_k}{\partial x_j}\right)\right\}$$

$$= \frac{1}{2}\left\{\delta_{ij} - \delta_{ki}\delta_{kj} + \frac{\partial u_k}{\partial x_i}\delta_{kj} + \frac{\delta u_k}{\partial x_j}\delta_{ki} - \frac{\delta u_k}{\delta x_i}\frac{\delta u_k}{\delta x_j}\right\}$$

$$= \frac{1}{2}\left\{\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right\} \quad \ldots\ldots (4.4)$$

Again $dx_i\,\delta x_i - dX_i\,\delta X_i = \delta_{ij}\,dx_i\,\delta x_j - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}dx_i\delta x_j$

$$\text{or, } \frac{dx_i}{dl}\frac{\delta x_i}{\delta l} - \frac{dX_i}{dL}\frac{\delta X_i}{\delta L}\frac{dL}{dl}\frac{\delta L}{\delta l} = 2\eta_{ij}\frac{dx_i}{dl}\frac{\delta x_j}{\delta l} = 2\eta_{ij}n_i m_j$$

$$\text{or, } \cos\theta - \frac{dL}{dl}\frac{\delta L}{\delta l}\cos\Theta = 2\eta_{ij}n_i m_j \quad (4.5)$$

Now, if $\eta_{ij} = 0$, $dL = dl$, $\delta L = \delta l$, and hence $\theta = \Theta$. This corresponds to rigid body deformation. Thus, the necessary and sufficient ceondition for rigid deformation at each point is $\eta_{ij} = 0$. The non-zero tensor η_{ij} gives rise to strain deformation, and it can be regarded as the measure of this strain deformatoin. It is easy to see that $\eta_{ij} = \eta_{ji}$, that is, η_{ij} is a symmetric tensor of rank two. It is called Eulerian finite strain tensor.

3.5. INFINITESIMAL STRAIN TENSOR :

In the case of small deformation one can obtain infinitesimal strain tensors from the finite strain tensors of Lagrangian and Eulerian methods through approximations. Let E_{ij} and e_{ij} be the Lagrangian and Eulerian linear strain tensors respectively defined by

$$E_{ij} = \frac{1}{2}\left\{\frac{\partial u_i(X_1, X_2, X_3)}{\partial X_j} + \frac{\partial u_j(X_1, X_2, X_3)}{\partial X_i}\right\} \quad \ldots\ldots (5.1)$$

and $\quad e_{ij} = \frac{1}{2}\left\{\frac{\partial u_i(x_1, x_2, x_3)}{\partial x_j} + \frac{\partial u_j(x_1, x_2, x_3)}{\partial x_i}\right\}$ ...... (5.2)

In fact, if we assume that all displacement gradients $\frac{\partial u_i}{\partial X_j}$, (i, j = 1, 2, 3), are numerically small compared to unity,

$$\text{i.e.,} \quad \left|\frac{\partial u_i}{\partial X_j}\right| << 1 \quad (i, j = 1, 2, 3) \quad ...... (5.3)$$

such that their squares and products can be neglected, then to the first order in the displacement gradients

$$r_{ij}(X_1, X_2, X_3) = E_{ij}(X_1, X_2, X_3) \quad ...... (5.4)$$

Also, we have $\frac{dl^2 - dL^2}{dL^2} = 2E_{ij}N_iN_j$ ...... (5.5)

and $\frac{dl}{dL}\frac{\delta l}{\delta L}\cos\theta - \cos\Theta = 2E_{ij}N_iN_j$ ...... (5.6)

Similarly, for $\left|\frac{\partial u_i}{\partial x_j}\right| << 1 \quad (i, j = 1, 2, 3)$ ...... (5.7)

we can have $\eta_{ij} = e_{ij}$ ...... (5.8)

$$\frac{dl^2 - dL^2}{dl^2} = 2e_{ij}n_in_j \quad ...... (5.9)$$

$$\cos\theta - \frac{dL}{dl}\cdot\frac{\delta L}{\delta l}\cos\Theta = 2e_{ij}n_im_j \quad ...... (5.10)$$

For small deformation, the displacements u_i are small, and the product terms like $u_i \frac{\partial u_j}{\partial X_i}$ can be neglected. Consequently, we have, since $x_i = X_i + u_i$,

$$u_i(x_1, x_2, x_3) = u_i(X_1 + u_1, X_2 + u_2, X_3 + u_3)$$

$$= u_i(X_1, X_2, X_3) + u_j\frac{\partial u_i}{\partial X_j} + ... \text{ (by Taylor expansion)}$$

$$= u_i(X_1, X_2, X_3)$$

Therefore, $$\frac{\partial u_i(X_1,X_2,X_3)}{\partial X_j} \simeq \frac{\partial u_i(x_1,x_2,x_3)}{\partial X_j}$$

$$= \frac{\partial u_i(x_1,x_2,x_3)}{\partial x_k}\frac{\partial x_k}{\partial X_j}$$

$$= \frac{\partial u_i(x_1,x_2,x_3)}{\partial x_k}\left(\delta_{kj} + \frac{\partial u_k}{\partial X_j}\right)$$

By neglecting the product term we have

$$\frac{\partial u_i(X_1,X_2,X_3)}{\partial X_j} \simeq \frac{\partial u_i(x_1 x_2,x_3)}{\partial x_k}\delta_{kj}$$

$$= \frac{\partial u_i(x_1,x_2,x_3)}{\partial x_j} \qquad \ldots\ldots \quad (5.11)$$

Therefore, we have

$$E_{ij} = \frac{1}{2}\left[\frac{\partial u_i(X_1,X_2,X_3)}{\partial X_j} + \frac{\partial u_j(X_1,X_2,X_3)}{\partial X_i}\right]$$

$$\simeq \frac{1}{2}\left[\frac{\partial u_i(x_1,x_2,x_3)}{\partial x_j} + \frac{\partial u_j(x_1,x_2,x_3)}{\partial x_i}\right]$$

$$= e_{ij} \qquad \ldots\ldots \quad (5.12)$$

Thus, we see that the Lagrangian and Eulerian linear strain tensors are identical, component by component, for small deformation when both the displacements gradients are small quantities such that their squares and products can be neglected. Therefore we do not require any distinction between the infinitesimal strain tensors in the two methods of description, and simply call it infinitesimal or small strain tensor.

Ex. 5.1. If the equations characterizing the deformation are given by

$$x_1 = X_1 + \in X_2$$
$$x_2 = X_2 - \in X_1 + \in X_3$$
$$x_3 = X_3 - \in X_2$$

determine the Lagragian and Eulerian finite strain tensors. Also find infinitesimal strain tensor when $\in$ is a small parameter.

Ans. $u_1 = x_1 - X_1 = \in X_2, \quad u_2 = x_2 - X_2 = -\in X_1 + \in X_3$

$u_3 = -\in X_2$

$$r_{11} = \frac{1}{2}\left[2\frac{\partial u_1}{\partial X_1} + \frac{\partial u_k}{\partial X_1}\frac{\partial u_k}{\partial X_1}\right] = \frac{1}{2}\left[0 + 0 + (-\in)^2 + 0\right] = \frac{\in^2}{2}$$

$$r_{22} = \frac{1}{2}\left[2\frac{\partial u_2}{\partial X_2} + \frac{\partial u_k}{\partial X_2}\frac{\partial u_k}{\partial X_2}\right] = \frac{1}{2}\left[0 + \in^2 + 0 + (-\in)^2\right] = \in^2$$

$$r_{33} = \frac{1}{2}\left[2\frac{\partial u_3}{\partial X_3} + \frac{\partial u_k}{\partial X_3}\frac{\partial u_k}{\partial X_3}\right] = \frac{1}{2}\left[0 + 0 + \in^2 + 0\right] = \frac{\in^2}{2}$$

$$r_{12} = \frac{1}{2}\left[\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \frac{\partial u_k}{\partial x_1}\frac{\partial u_k}{\partial X_2}\right] = \frac{1}{2}\left[\in - \in + 0 + 0 + 0\right] = 0$$

$$r_{23} = \frac{1}{2}\left[\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} + \frac{\partial u_k}{\partial X_2}\frac{\partial u_k}{\partial X_3}\right] = \frac{1}{2}\left[\in - \in + 0 + 0 + 0\right] = 0$$

$$r_{31} = \frac{1}{2}\left[\frac{\partial u_3}{\partial X_1} + \frac{\partial u_1}{\partial X_3} + \frac{\partial u_k}{\partial X_1}\frac{\partial u_k}{\partial X_3}\right] = \frac{1}{2}\left[0 + 0 + 0 + (-\in)\in + 0\right]$$

$$= -\frac{\in^2}{2}$$

Therefore, $$\left(r_{ij}\right) = \begin{pmatrix} \in^2/2 & 0 & -\in^2/2 \\ 0 & \in^2 & 0 \\ -\in^2/2 & 0 & \in^2/2 \end{pmatrix}$$

$$\eta_{23} = \frac{1}{2}\left[\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} - \frac{\partial u_k}{\partial x_2}\frac{\partial u_k}{\partial x_3}\right] = \frac{1}{2}\left[\frac{\in}{1 + 2\in^2} + \frac{(-\in)}{1 + 2\in^2} - \frac{\in\left(-\in^2\right)}{\left(1 + 2\in^2\right)^2} - \frac{\left(2\in^2\right)\in}{\left(1 + 2\in^2\right)^2} - \frac{(-\in)\left(\in^2\right)}{\left(1 + 2\in^2\right)^2}\right]$$

$$= \frac{1}{2}\left[\frac{\epsilon^3}{\left(1+2\epsilon^2\right)^2} - \frac{2\epsilon^3}{\left(1+2\epsilon^2\right)^2} + \frac{\epsilon^3}{\left(1+2\epsilon^2\right)^2}\right] = 0$$

$$\eta_{31} = \frac{1}{2}\left[\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} - \frac{\partial u_k}{\partial x_1}\frac{\partial u_k}{\partial x_3}\right] = \frac{1}{2}\left[\frac{-\epsilon^2}{1+2\epsilon^2} + \frac{-\epsilon^2}{1+2\epsilon^2} - \frac{\epsilon^2\left(-\epsilon^2\right)}{\left(1+2\epsilon^2\right)^2}\right.$$

$$\left. - \frac{(-\epsilon)\epsilon}{\left(1+2\epsilon^2\right)^2} - \frac{\left(-\epsilon^2\right)\epsilon^2}{\left(1+2\epsilon^2\right)^2}\right]$$

$$= \frac{1}{2\left(1+2\epsilon^2\right)^2}\left[-2\epsilon^2\left(1+2\epsilon^2\right) + \epsilon^4 + \epsilon^2 + \epsilon^4\right]$$

$$= \frac{1}{2\left(1+2\epsilon^2\right)^2}\left[-2\epsilon^4 - \epsilon^2\right] = -\frac{\epsilon^2}{2\left(1+2\epsilon^2\right)}$$

Therefore,

$$\left(n_{ij}\right) = \frac{1}{1+2\epsilon^2}\begin{pmatrix} \epsilon^2/2 & 0 & -\epsilon^2/2 \\ 0 & \epsilon^2 & 0 \\ -\epsilon^2/2 & 0 & \epsilon^2/2 \end{pmatrix}$$

For small ϵ, the infinitesimal strain tensor is zero (to the first order of ϵ). This corresponds to rigid deformation.

Ex.5.2. If the equations characterizing the deformation are given by

$$x_1 = X_1 + \epsilon X_2$$

$$x_2 = X_2 + \epsilon X_3$$

$$x_3 = X_3 + \epsilon X_2, \quad \text{where } \epsilon \text{ is small,}$$

determine the infinitesimal strain tensor.

Ans.

$$u_1 = x_1 - X_1 = \epsilon X_2$$

$$u_2 = x_2 - X_2 = \epsilon X_3$$

$$u_3 = x_3 - X_3 = \epsilon X_2$$

$$E_{ij} = \epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right],\ E_{11} = \frac{\partial u_1}{\partial X_1} = 0,\ E_{22} = \frac{\partial u_2}{\partial X_2} = 0$$

$$E_{33} = \frac{\partial u_3}{\partial X_3} = 0,\ E_{12} = \frac{1}{2}\left[\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right] = \frac{1}{2}[\epsilon + 0] = \frac{\epsilon}{2}$$

$$E_{23} = \frac{1}{2}\left[\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\right] = \frac{1}{2}[\epsilon + \epsilon] = \epsilon$$

$$E_{31} = \frac{1}{2}\left[\frac{\partial u_3}{\partial X_1} + \frac{\partial u_1}{\partial X_3}\right] = \frac{1}{2}[0 + 0] = 0$$

Now, solving for X_1, X_2, X_3 we have

$$X_1 = \frac{x_1\left(1+\epsilon^2\right) - \epsilon x_2 + \epsilon^2 x_3}{1+2\epsilon^2}$$

$$X_2 = \frac{\epsilon x_1 + x_2 - \epsilon x_3}{1+2\epsilon^2}$$

$$X_3 = \frac{\epsilon^2 x_1 + \epsilon x_2 + (1+\epsilon^2)x_3}{1+2\epsilon^2}$$

Therefore, $u_1 = x_1 - X_1 = \dfrac{\epsilon^2 x_1 + \epsilon x_2 - \epsilon^2 x_3}{1+2\epsilon^2}$

$$u_2 = x_2 - X_2 = \frac{2\epsilon^2 x_2 - \epsilon x_1 + \epsilon x_3}{1+2\epsilon^2}$$

$$u_3 = x_3 - X_3 = \frac{\epsilon^2 x_3 - \epsilon^2 x_1 - \epsilon x_2}{1+2\epsilon^2}$$

$$\eta_{11} = \frac{1}{2}\left[2\frac{\partial u_1}{\partial x_1} - \frac{\partial u_k}{\partial x_1}\frac{\partial u_k}{\partial x_1}\right] = \frac{1}{2}\left[2\frac{\epsilon^2}{1+2\epsilon^2} - \left(\frac{\epsilon^2}{1+2\epsilon^2}\right)^2 - \left(\frac{-\epsilon}{1+2\epsilon^2}\right)^2 - \left(\frac{-\epsilon^2}{1+2\epsilon^2}\right)^2\right]$$

$$= \frac{1}{2}\left[2\frac{\epsilon^2}{1+2\epsilon^2} - \frac{2\epsilon^4}{\left(1+2\epsilon^2\right)^2} - \frac{\epsilon^2}{\left(1+2\epsilon^2\right)^2}\right] = \frac{1}{2\left(1+2\epsilon^2\right)^2}\left\{2\epsilon^2\left(1+2\epsilon^2\right) - 2\epsilon^4 - \epsilon^2\right\}$$

$$= \frac{1}{2\left(1+2\epsilon^2\right)^2}\left\{\epsilon^2 + 2\epsilon^4\right\} = \frac{\epsilon^2}{2\left(1+2\epsilon^2\right)}$$

$$\eta_{22} = \frac{1}{2}\left[2\frac{\partial u_2}{\partial x_2} - \frac{\partial u_k}{\partial x_2}\frac{\partial u_k}{\partial x_2}\right] = \frac{1}{2}\left[2\frac{2\epsilon^2}{1+2\epsilon^2} - \frac{\epsilon^2}{\left(1+2\epsilon^2\right)^2} - \frac{4\epsilon^4}{\left(1+2\epsilon^2\right)^2} - \frac{\epsilon^2}{\left(1+2\epsilon^2\right)^2}\right]$$

$$= \frac{1}{2\left(1+2\epsilon^2\right)^2}\left\{4\epsilon^2\left(1+2\epsilon^2\right) - 2\epsilon^2 - 4\epsilon^4\right\} = \frac{1}{2\left(1+2\epsilon^2\right)^2}\left\{2\epsilon^2 + 4\epsilon^4\right\}$$

$$= \frac{\epsilon^2}{1+2\epsilon^3}$$

$$\eta_{33} = \frac{1}{2}\left[2\frac{\partial u_3}{\partial x_3} - \frac{\partial u_k}{\partial x_3}\frac{\partial u_k}{\partial x_3}\right] = \frac{1}{2}\left[2\frac{\epsilon^2}{1+2\epsilon^2} - \frac{\epsilon^4}{\left(1+2\epsilon^2\right)^2} - \frac{\epsilon^2}{\left(1+2\epsilon^2\right)^2} - \frac{\epsilon^4}{\left(1+2\epsilon^2\right)^2}\right]$$

$$= \frac{1}{2\left(1+2\epsilon^2\right)^2}\left\{2\epsilon^2\left(1+2\epsilon^2\right) - 2\epsilon^4 - \epsilon^2\right\}$$

$$= \frac{1}{2\left(1+2\epsilon^2\right)^2}\left(\epsilon^2 + 2\epsilon^4\right) = \frac{\epsilon^2}{2\left(1+2\epsilon^2\right)}$$

$$\eta_{12} = \frac{1}{2}\left[\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} - \frac{\partial u_k}{\partial x_1}\frac{\partial u_k}{\partial x_2}\right] = \frac{1}{2}\left[\frac{\epsilon}{1+2\epsilon^2} - \frac{\epsilon}{1+2\epsilon^2} - \frac{\epsilon^2.\epsilon}{\left(1+2\epsilon^2\right)^2}\right.$$

$$\left. - \frac{(-\epsilon)\left(2\epsilon^2\right)}{\left(1+2\epsilon^2\right)^2} - \frac{\left(-\epsilon^2\right)(-\epsilon)}{\left(1+2\epsilon^2\right)^2}\right]$$

$$= \frac{1}{2\left(1+2\epsilon^2\right)^2}\left[-\epsilon^3 + 2\epsilon^3 - \epsilon^3\right] = 0$$

Therefore $\left(E_{ij}\right) = \left(e_{ij}\right) = \begin{pmatrix} 0 & \epsilon/2 & 0 \\ \epsilon/2 & 0 & \epsilon \\ 0 & \epsilon & 0 \end{pmatrix}$

3.6. GEOMETRIC INTERPRETATION OF STRAIN COMPONENTS

We shall follow the Lagrangian description for deformation in order to give geometric interpretation of strain components. We have already considered the change of length dL of a material line element P_0O_0 (oriented in the direction of (N_1, N_2, N_3)) in the undeformed state into the length dl of the line element PQ of orientatión in the direction (n_1, n_2, n_3) in the deformed state. It is given by

$$\frac{dl^2 - dL^2}{dL^2} = 2E_{ij}N_iN_j \quad \ldots\ldots \quad (6.1)$$

with $E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$ being the infinitesimal strain tensor at P_0. We can write (6.1) as

$$\frac{dl^2}{dL^2} = 1 + 2E_{ij}N_iN_j$$

or, $$\frac{dl}{dL} = \left(1 + 2E_{ij}N_iN_j\right)^{1/2} \doteq 1 + \frac{1}{2}.2E_{ij}N_iN_j + \ldots\ldots$$

If the strain components are so small, such that one can neglect their squares and products, then we have

$$\frac{dl}{dL} = 1 + E_{ij}N_iN_j$$

or, $$\frac{dl - dL}{dL} = E_{ij}N_iN_j \quad \ldots\ldots (6.2)$$

Now, $\frac{dl - dL}{dL}$ is the extension per unit original length of the line element oriented in the direction (N_1, N_2, N_3). It is called small extensional strain, and is denoted by $E_{(N)}$. Therefore,

$$E_{(N)} = Eij\ Ni\ Nj \quad \ldots\ldots (6.3)$$

Now, if the line element was initially parallel to X_1 axis, that is $N_1 = 1, N_2 = N_3 = 0$, we get $E_{(1)} = E_{11}$. Thus, E_{11} is the extension per unit original length of a line element initially oriented in the direction parallel to X_1 axis. Similarly, E_{22} and E_{33} represent, respectively, the extensions of the line elements per unit original length, which are initially parallel to X_2 and X_3 axes. E_{11}, E_{22}, E_{33} are called normal or extensional strain.

Now, if the material lines P_0Q_0 and P_0R_0 are orthogonal to each other in the undeformed state, that is $\Theta = \pi/2$, then we have from (5.6).

$$\frac{dl}{dL}\cdot\frac{\delta l}{\delta L}\cos\theta - \cos\frac{\pi}{2} = 2E_{ij}N_iM_j \qquad (6.4)$$

or, $$\sin\left(\frac{\pi}{2}-\theta\right) = \frac{2E_{ij}N_iM_j}{\frac{dl}{dL}\cdot\frac{\delta l}{\delta L}} \qquad (6.5)$$

The right angle between the two orthogonal material lines in the undeformed state is, thus, decreased by an angle $\frac{\pi}{2}-\theta$, which is called shear along the two lines. Let $\gamma_{(NM)}$ be this shear along two orthogonal line elements initially oriented in the direction (N_1, N_2, N_3) and (M_1, M_2, M_3), that is, $\gamma_{(NM)} = \frac{\pi}{2}-\theta$.

Therefore, $$\sin\gamma_{(NM)} = \frac{2E_{ij}N_iM_j}{\frac{dl}{dL}\frac{\delta l}{\delta L}} \quad \ldots (6.6)$$

If E_1 and E_2 are the extensions of P_0Q_0 and P_0R_0 respectively then we have

$$\left.\begin{aligned} \frac{dl-dL}{dL} &= E_1 \text{ or, } \frac{dl}{dL} = 1+E_1 \\ \frac{\delta l-\delta L}{\delta L} &= E_2, \text{ or, } \frac{\delta l}{\delta L} = 1+E_2 \end{aligned}\right\} \quad \ldots\ldots(6.7)$$

Therefore from (6.6) and (6.7) we have

$$\sin\gamma_{(NM)} = \frac{2E_{ij}N_iM_j}{(1+E_1)(1+E_2)}$$

For small deformation, $\sin\gamma_{(NM)} \simeq \gamma_{(NM)}$ and we have

$$\gamma_{(NM)} = \frac{2E_{ij}\,N_i\,M_j}{1+E_1+E_2+E_1E_2} = 2E_{ij}\,N_i\,M_j(1+E_1+E_2+E_1E_2)^{-1}$$

$$\simeq 2E_{ij}\,N_i\,M_j \qquad (6.8)$$

If the pair of orthogonal line elements initially parallel to X_2 and X_3 axes respectively, then $N_1 = 0$, $N_2 = 1$, $N_3 = 0$, and $M_1 = 0$, $M_2 = 0$, $M_3 = 1$. Consequently, $\gamma_{(23)} = 2E_{23}$ or, $E_{23} = \frac{1}{2}\gamma_{(23)}$.

E_{23} represent, thus, one-half of the shear between two line elements initially parallel to X_2 and X_3 axes. Similar interpretation holds for E_{31} and E_{12}. E_{23}, E_{31}, E_{12} are called shearing strains. E_{ij} thus denote increase in length of a line element per unit original length or decrease in right angle between two line elements. For rigid deformation $E_{ij} = 0$.

3.7. CHANGE IN VOLUME DUE TO DEFORMATION

Let us consider an elementary rectangular parallelopiped at the piont $P_0(X_1, X_2, X_3)$, in the undeformed state, having edges of lengths dX_1, dX_2, dX_3 parallel to the co-ordinates axes. The volume element of this parallelopiped with one of its vertices at P_0 is given as

$$dV_0 = dX_1\, dX_2\, dX_3 \qquad (7.1)$$

The position vector of P_0 is $\vec{X} = (X_1, X_2, X_3)$ and it moves to the point P in the deformed state. Let the position vector of P be $\vec{x} = (x_1, x_2, x_3)$. In the deformed state the position vectors of the other vertices of the parallelopiped are

$$\vec{x}+d\vec{x}^{(1)},\ \vec{x}+d\vec{x}^{(2)},\ \vec{x}+d\vec{x}^{(3)}$$

The volume element of the parallelopiped in the deformed state then will be

$$dV = d\vec{x}^{(1)}.\left(d\vec{x}^{(2)} \times d\vec{x}^{(3)}\right) \qquad (7.2)$$

or, we can write

$$dV = \epsilon_{ijk}.dx_i^{(1)}.dx_j^{(2)}\, dx_k^{(3)} \qquad (7.3)$$

where ϵ_{ijk}, the alternating symbol, is defined by

$$\left.\begin{aligned} \epsilon_{ijk} &= 0, \text{ if any two of } i, j, k \text{ are equal} \\ &= 1, \text{ if } i, j, k \text{ are even permutation of } 1,2,3 \\ &= -1, \text{if } i, j, k \text{ are odd permutation of } 1,2,3 \end{aligned}\right\} \qquad (7.4)$$

In Lagrangian description of deformation

$$x_i = x_i(X_1, X_2, X_3) \qquad (7.5)$$

$$\therefore\ dx_i = \frac{\partial x_i}{\partial X_j} dX_j \qquad (7.6)$$

Since the material line elements dX_1, dX_2, dX_3 of the undeformed state become the line elements $dx_i^{(1)}$, $dx_i^{(2)}$, $dx_i^{(3)}$ in the deformed state, we have

$$\left.\begin{aligned} dx_i^{(1)} &= \frac{\partial x_i}{\partial X_1} dX_1 \\ dx_i^{(2)} &= \frac{\partial x_i}{\partial X_2} dX_2 \\ dx_i^{(3)} &= \frac{\partial x_i}{\partial X_3} dX_3 \end{aligned}\right\} \qquad (7.7)$$

Subsitituting (7.7) into (7.3), we get

$$dV = J\, dV_0$$

where $$J = \epsilon_{ijk} \frac{\partial x_i}{\partial X_1} \frac{\partial x_j}{\partial X_2} \frac{\partial x_k}{\partial X_3} = \left|\frac{\partial x_i}{\partial X_j}\right|$$

$$= \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_2}{\partial X_1} & \frac{\partial x_3}{\partial X_1} \\ \frac{\partial x_1}{\partial X_2} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_3}{\partial X_2} \\ \frac{\partial x_1}{\partial X_3} & \frac{\partial x_2}{\partial X_3} & \frac{\partial x_3}{\partial X_3} \end{vmatrix} = \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} \qquad (7.8)$$

Again, $$J = \left|\frac{\partial x_i}{\partial X_j}\right| = \left|\frac{\partial(X_i + u_i)}{\partial X_j}\right| = \left|\delta_{ij} + \frac{\partial u_i}{\partial X_j}\right|.$$

$$\simeq 1 + \frac{\partial u_i}{\partial X_i}, \quad \text{for small strain}$$

$$= 1 + E_{11} + E_{22} + E_{23}$$

Therefore, $$E_{11} + E_{22} + E_{33} = J - 1 = \frac{dV}{dV_0} - 1 = \frac{dV - dV_0}{dV_0} \qquad (7.9)$$

Thus, for small strain deformation $E_{11} + E_{22} + E_{33}$ represents the change in volume per unit original volume, and is called dilatation or volumetric strain.

Unit : 4 ❑ Analysis of Strain

4.1. RELATIVE DISPLACEMENT

If the two neighbouring material points $P_0(X_1, X_2, X_3)$ and $Q_0(X_1 + dX_1, X_2 + dX_2, X_3 + dX_3)$ of the undeformed state experience displacements $\vec{u} = (u_1, u_2, u_3)$ and $\vec{u} + d\vec{u} = (u_1 + du_1, u_2 + du_2, u_3 + du_3)$ respectively, then $d\vec{u}$ is the relative displacement of the material point Q_0 with respect to the material point P_0. In the Lagrangian description, u_i must be functions of X_1, X_2, X_3, that is,

$$u_i = f_i(X_1, X_2, X_3) \qquad \text{...... (1.1)}$$

Thus, $u_i + du_i = f_i(X_1 + dX_2, X_2 + dX_2, X_3 + dX_3)$

$$= f_i(X_1, X_2, X_3) + \frac{\partial f_i}{\partial X}dX_1 + \frac{\partial f_i}{\partial X_2}dX_2 + \frac{\partial f_i}{\partial X_3}dX_3 + \ldots$$

Since the points P_0 and Q_0 are neighbouring points very closed to each other, dX_i must be small. Therefore, neglecting the higher powers of dX_i in the above Taylor's expansion, we have

$$u_i + du_i = f_i(X_1, X_2, X_3) + \frac{\partial f_i}{\partial X_1}dX_1 + \frac{\partial f_i}{\partial X_2}dX_2 + \frac{\partial f_i}{\partial X_3}dX_3$$

$$= u_i + \frac{\partial u_i}{\partial X_j}dX_j$$

or, $$du_i = \frac{\partial u_i}{\partial X_j}dX_j \qquad \text{......(1.2)}$$

We can write

$$\frac{\partial u_i}{\partial X_j} = R_{ij} + E_{ij} \qquad \text{......(1.3)}$$

with $E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) = E_{ji}$, which is the symmetric small strain tensor of order 2,

and $$R_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_i} + \frac{\partial u_i}{\partial X_i}\right) = -R_{ji} \qquad \text{......(1.4)}$$

Here, R_{ij} is a skew symmetric tensor of order 2.

Therefore, from (1.2) and (1.3) we get

$$du_i = R_{ij}\, dX_j + E_{ij}\, dX_j \qquad ......(1.5)$$

We form a vector R_i by setting

$$R_i = \epsilon_{ijk} R_{kj}$$

$$\epsilon_{ijk} R_i = \epsilon_{ijk} \epsilon_{ipq} R_{qp} = (\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp})R_{qp}$$

$$= R_{kj} - R_{jk} = 2R_{kj} \quad (\because R_{jk} = -R_{kj})$$

Therefore, $R_{kj} = \frac{1}{2} \epsilon_{ijk} R_i$

which is the inverse relation of (1.6).

Now, $R_{ij}\, dX_j = \frac{1}{2} \epsilon_{kji} R_k\, dX_j$ (using (1.7))

$$= \frac{1}{2}\left(\vec{R} \times d\vec{X}\right)_i \qquad ......(1.8)$$

where $\vec{R} = (R_1, R_2, R_3)$ and $d\vec{X} = \overrightarrow{P_0Q_0} = (dX_1, dX_2, dX_3)$

Now, $R_i = \epsilon_{ijk} R_{kj} = \frac{\epsilon_{ijk}}{2}\left(\frac{\partial u_k}{\partial X_j} - \frac{\partial u_j}{\partial X_k}\right) = \frac{1}{2}\left(\epsilon_{ijk}\frac{\partial u_k}{\partial X_j} - \epsilon_{ijk}\frac{\partial u_j}{\partial X_k}\right)$

$$= \frac{1}{2}\left(\epsilon_{ijk}\frac{\partial u_k}{\partial X_j} - \epsilon_{ikj}\frac{\partial u_k}{\partial X_j}\right) = \frac{1}{2}\left(\epsilon_{ijk}\frac{\partial u_k}{\partial X_j} + \epsilon_{ijk}\frac{\partial u_k}{\partial X_j}\right)$$

$$= \epsilon_{ijk}\frac{\partial u_k}{dX_j} = (\text{rot } \vec{u})_i \quad \text{where } \vec{u} = (u_1, u_2, u_3)$$

That is, $\vec{R} = \text{rot } \vec{u}$ ......(1.9)

Therefore, the 1st term of the R.H.S. of (1.5) represents a relative displacement involving small rigid body rotation of the neighbourhood element of P through an angle $\frac{1}{2}\vec{R} = \frac{1}{2}\text{rot } \vec{u}$. The vector $\vec{R}$ is called small rotation vector and the tensor R_{ij} is the small rolation tenser. This first term of R.H.S. of (1.5) can not make any strain deformation which is caused only by the second term in the relation of the relative displacement given in (8.5).

Thus, if the deformation consists of strain deformation only (that is, there is no rigid body deformation), then the relative displacement is given by

$$du_i = E_{ij}\, dX_j \qquad \text{......(1.10)}$$

Now, the material line element P_0Q_0 in the undeformed state changes to the line element PQ. Also $\overrightarrow{P_0Q_0} = (dX_1, dX_2, dX_3)$ and $\overrightarrow{PQ} = (dx_1, dx_2, dx_3)$. We know that

$$dx_i = du_i + dX_i \quad \text{or,} \quad d\vec{x} = d\vec{u} + d\vec{X}$$

The strain vector at P_0 can be defined by

$$\vec{E}^{(N)} = \frac{\overrightarrow{PQ} - \overrightarrow{P_0Q_0}}{\left|\overrightarrow{P_0Q_0}\right|} \qquad \text{......(1.11)}$$

$$\text{or,} \quad \vec{E}^{(N)} = \frac{d\vec{x} - d\vec{X}}{\left|d\vec{X}\right|} = \frac{d\vec{x} - d\vec{X}}{dL}$$

$$\text{or,} \quad E_i^{(N)} = \frac{dx_i - dX_i}{dL} = \frac{du_i}{dL} \qquad \text{......(1.12)}$$

$$= \frac{E_{ij}dX_j}{dL} \quad \text{(using (1.10))}$$

$$= E_{ij}\, N_j \qquad \text{......(1.13)}$$

This gives a relation between strain vector and strain tensor at P_0. Normal component of strain vector $\vec{E}^{(N)}$ in the direction (N_1, N_2, N_3) is given by

$$E_i^{(N)} N_i = E_{ij}N_jN_i = E_{(N)}$$

$$= E_{11} \text{ when } N_1 = 1,\ N_2 = 0 = N_3$$

$$= E_{22} \text{ when } N_2 = 1,\ N_1 = 0 = N_3$$

$$= E_{33} \text{ when } N_3 = 1,\ N_1 = 0 = N_2$$

For this reason extensional strain E_{11}, E_{22}, E_{33} are also called normal strain.

4.2. STRAIN QUADRIC

We consider a geometric treatment in order to understand the state of deformation in the neighbourhood of a point in the underformed state of a continuum body.

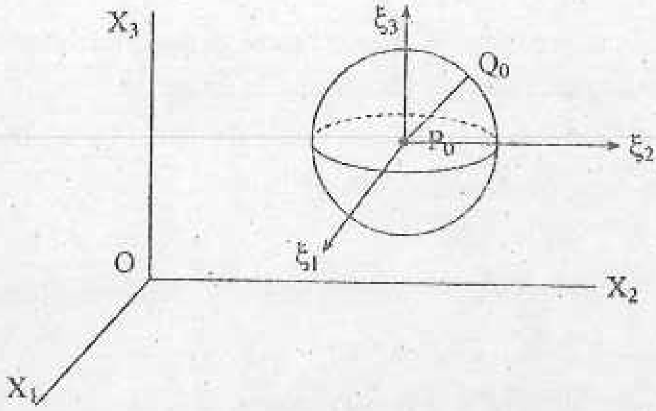


Fig. 2

Let $P_0(X_1, X_2, X_3)$ be a point in the undeformed state of the continum body. The axes OX_1, OX_2, OX_3 are fixed in space. Let E_{ij} be the small strain tensor at P_0. Taking P_0 as origin we can also take a local system of axes $P_0\xi_1$, $P_0\xi_2$, $P_0\xi_3$ being parallel to the axes OX_1, OX_2, OX_3 respectively. Then quadric surface with its centre at P_0 given by

$$E_{ij}\,\xi_i\,\xi_j = 1 \qquad \ldots\ldots(2.1)$$

is called the "strain quadric".

Now, let us draw any line P_0Q_0 through the centre P_0 to intersect the quadric surface (2.1) at the point Q_0. Let L be the length of P_0Q_0 and (N_1, N_2, N_3) be the direction cosines of P_0Q_0. Also let (ξ_1, ξ_2, ξ_3) be the coordinates of Q_0, and $E_{(N)}$ be the extension of the line element P_0Q_0 in the direction of P_0Q_0. Then we know that

$$E_{(N)} = E_{ij}\,N_i\,N_j \qquad \ldots\ldots(2.2)$$

Also, $$N_i = \frac{\xi_i}{L} \qquad \ldots\ldots(2.3)$$

Therefore, we have

$$E_{(N)} = \frac{E_{ij}\xi_i\xi_j}{L^2}$$

Since Q_0 lies on the strain quadric, (ξ_1, ξ_2, ξ_3) must satisfy (2.1). Consequently we have

$$E_{(N)} = \frac{1}{L^2} \qquad \text{...... (2.4)}$$

That is, the extension of any line element through the centre of strain quadric along its direction is equal to the inverse of the square of its length.

Now, let $\bar{u}_i$ be the displacement of the material point at Q_0 relative to the P_0 due to strain deformation only. Then from (1.10) we have

$$\bar{u}_i = E_{ij}\xi_j \qquad \text{......(2.5)}$$

since (ξ_1, ξ_2, ξ_3) are the relative coordinates of Q_0 with respect to P_0.

Writing $E_{ij}\xi_i\xi_j = 2G(\xi_1, \xi_2, \xi_3)$ ......(2.6)

we can now have the equation of strain quadric (2.1) as

$$2G(\xi_1, \xi_2, \xi_3) = 1 \qquad \text{......(2.7)}$$

From (2.6), we have

$$\frac{\partial}{\partial \xi_i}(2G) = \frac{\partial}{\partial \xi_i}(E_{k\ell}\xi_k\xi_\ell) = E_{k\ell}\left(\frac{\partial \xi_k}{\partial \xi_i}\xi_\ell + \frac{\partial \xi_\ell}{\partial \xi_i}\xi_k\right)$$

$$= E_{k\ell}(\delta_{ki}\xi_\ell + \delta_{\ell i}\xi_k) = E_{i\ell}\xi_\ell + E_{ki}\xi_k$$

$$= E_{ij}\xi_j + E_{ji}\xi_j = E_{ij}\xi_j + E_{ij}\xi_j = 2E_{ij}\xi_j$$

$$\therefore \quad \frac{\partial G}{\partial \xi_i} = E_{ij}\xi_j = \bar{u}_i \quad \text{(using (2.5))} \qquad \text{......(2.8)}$$

Since $\frac{\partial G}{\partial \xi_i}$ are direction ratios of the normal to the quadric surface (2.7) at the point Q_0, it follows from (2.8) that the relative displcement at any point on the strain quadric to that at the centre is directed along the normal to the quadric at that point.

4.3. PRINCIPAL STRAIN

If the direction of a line element at a given point of a continuum body remains unchanged by strain deformation then this direction is known as the principal direction of strain or principal axis of strain. The extension occuring along the principal direction is called principal strain.

Let $P_0(X_1, X_2, X_3)$ and $Q_0(X_1 + dX_1, X_2 + dX_2, X_3 + dX_3)$ be two neighbouring material points (see Fig. 1). Let the line element P_0Q_0 be of length dL oriented in the direction (N_1, N_2, N_3), Then

$$N_i = \frac{dX_i}{dL} \quad i = 1, 2, 3 \qquad \text{......(3.1)}$$

$$N_i N_i = 1$$

In the deformed state P_0 and Q_0 move to the points $P(x_1, x_2, x_3)$ and $Q(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$ respectively. Also u_i and $u_i + du_i$ $(i = 1, 2, 3)$ are the displacements of P_0 and Q_0 respectively. Let E_{ij} be the strain tensor at P_0.

Now, if the line element is the principal direction of strain at P_0 then its direction must remain unchanged due to strain deformation, that is, PQ must be parallel to P_0Q_0. For this, we must have

$$du_i \ \alpha \ dX_i \quad i = 1, 2, 3$$

(that is, the relative displacement vector is proportional to $\overrightarrow{P_0Q_0}$)

or, we can write

$du_i = EdX_i$ where E is the constant of proportionality

or, $E = \dfrac{du_i}{dX_i} = \dfrac{dx_i - dX_i}{dX_i}$ (using (3.13) of unit 3)

= extension of the component dX_i per unit length, that is, the extension of the line element P_0Q_0 in the direction of P_0Q_0.

E is, thus, the principal strain.

Again strain vector $E_i^{(N)} = \dfrac{du_i}{dL} = \dfrac{du_i}{dX_i}\dfrac{dX_i}{dL} = E\dfrac{dX_i}{dL}$

$= EN_i$ (using (3.1)) ......(3.2)

Also, we know that

$$E_i^{(N)} = E_{ij}N_j$$

$$\therefore \quad EN_i = E_{ij} N_j \qquad \text{......(3.3)}$$

$$\text{or,} \quad E\,\delta_{ij}\, N_j = E_{ij}\, N_j$$

$$\text{or,}\quad (E_{ij} - \delta_{ij}E)N_j = 0 \quad i = 1, 2, 3 \qquad \text{......(3.4)}$$

That is,

$$\left.\begin{aligned} (E_{11} - E)N_1 + E_{12}N_2 + E_{13}N_3 &= 0 \\ E_{21}N_1 + (E_{22} - E)N_2 + E_{23}N_3 &= 0 \\ E_{31}N_1 + E_{32}N_2 + (E_{33} - E)N_3 &= 0 \end{aligned}\right\} \text{......(3.5)}$$

This is a set of three homogeneous equations for N_1, N_2, N_3 which satisfy the condition (3.1), ie, $N_1^2 + N_2^2 + N_3^2 = 1$. The trivial solution $N_1 = N_2 = N_3 = 0$ of (3.5) does not satisfy (3.1), hence should be rejected. The condition for existence of nontrival non zero solution of (3.5) is that the determinant

$$\begin{vmatrix} E_{11} - E & E_{12} & E_{13} \\ E_{21} & E_{22} - E & E_{23} \\ E_{31} & E_{32} & E_{33} - E \end{vmatrix} = 0 \quad \text{......(3.6)}$$

This is a cubic equation for E, the principal strain. Thus equation is known as the characteristic equation. The roots of this equation are E_1, E_2, E_3 which one the three principal strains. With each of these roots one can solue (3.5) using (3.1) to find the corresponding principal axis of strain, ie., the direction cosines (N_1, N_2, N_3) of principal axis.

Example 3.1. Show that all principal strains are real.

Proof : Let us suppose that one root of the equation (3.6), say E_1, is complex. Then the complex conjugate E_1^* of E_1 must also be a root of this equation. Trom (3.3), we have

$$E_{ij}\ N_j^{(1)} = E_1\ \ N_i^{(1)} \qquad (i = 1, 2, 3) \text{......(3.7)}$$

where $N_i^{(1)}$ (i = 1, 2, 3) represent the direction cosines of the corresponding principal axis. Since E_{ij} are real, we have from the above equation by taking its complex conjugate,

$$E_{ij}\, N_j^{*(1)} = E_1^*\, N_i^{*(1)} \qquad \text{......(3.8)}$$

Multiplying the equations (3.7) and (3.8) by $N_i^{*(1)}$ and $N_i^{(1)}$ respectively, and summing over i we have

$$\left.\begin{aligned} E_{ij}\, N_j^{(1)} N_i^{*(1)} &= E_1\, N_i^{(1)}\, N_i^{*(1)} = E_1 \sum_i \left| N_i^{(1)} \right|^2 \\ \text{and} \quad E_{ij}\, N_j^{*(1)}\, N_i^{(1)} &= E_1^*\, N_i^{*(1)}\, N_i^{(1)} = E_1^* \sum_i \left| N_i^{(1)} \right|^2 \end{aligned}\right\} \quad \ldots\ldots(3.9)$$

Now, $E_{ij}\, N_j^{(1)}\, N_i^{*(1)} = E_{ji}\, N_i^{(1)}\, N_j^{*(1)}$ (interchanging dummy suffixes)

$= E_{ij}\, N_i^{(1)}\, N_j^{*(1)}$ (Since E_{ij} is symmetric)

Therefore, from (3.9) we get

$$E_1 \sum_i \left| N_i^{(1)} \right|^2 = E_1^* \sum_i \left| N_i^{(1)} \right|^2$$

Since $\sum_i \left| N_i^{(1)} \right|^2 \neq 0$ unless all $N_i^{(1)}$ (i = 1, 2, 3) are zero, we must have $E_1 = E_1^*$, that is E_1 is real.

Example 3.2. Show that the three principal direction of strain are mutually perpendicular if the corresponding three principal strains are distinct.

Proof : Let E_1, E_2, E_3 be three distinct roots of the cubic equation (3.6). These are the principal strains, and the direction cosines of the correspnding principal axes are given by $N_i^{(1)}, N_i^{(2)}, N_i^{(3)}$ (i = 1, 2, 3). We have, from (3.3),

$$E_{ij}\, N_j^{(1)} = E_1\, N_i^{(1)} \quad (i = 1, 2, 3) \qquad \ldots\ldots(3.10)$$

$$E_{ij}\, N_j^{(2)} = E_2\, N_i^{(2)} \quad (i = 1, 2, 3) \qquad \ldots\ldots(3.11)$$

$$E_{ij}\, N_j^{(3)} = E_3\, N_i^{(3)} \quad (i = 1, 2, 3) \qquad \ldots\ldots(3.12)$$

Multiplying the equations (3.10) and (3.11) by $N_i^{(2)}$ and $N_i^{(1)}$ respectively, and summing over i we have

$$E_{ij}\, N_j^{(1)}\, N_i^{(2)} = E_1\, N_i^{(1)}\, N_i^{(2)}$$

and $\quad E_{ij} N_j^{(2)} N_i^{(1)} = E_2 N_i^{(2)} N_i^{(1)}$

$$E_{ij} N_j^{(2)} N_i^{(1)} = E_{ji} N_i^{(2)} N_j^{(1)} \text{ (interchanging the dummy suffixes)}$$

$$= E_{ij} N_j^{(1)} N_i^{(2)} \quad (\text{since } E_{ij} = E_{ji})$$

Therefore, $E_1 N_i^{(1)} N_i^{(2)} = E_2 N_i^{(2)} N_i^{(1)}$

or, $(E_1 - E_2) N_i^{(1)} N_i^{(2)} = 0$

Since $E_1 \neq E_2$, we must have

$$N_i^{(1)} N_i^{(2)} = 0 \qquad \text{......(3.13)}$$

This shows that the directions $N_i^{(1)}$ and $N_i^{(2)}$ are orthogonal for the case of $E_1 \neq E_2$. Similar results can be obtained for other set of pair of roots. Hence we have that the principal directions of strain corresponding to distinct principal strains are orthogonal to each other.

4.4. STRAIN INVARIANTS

Expanding the determinant in (3.6) we have the following cubic equation for principal strain E :

$$E^3 - E^2\theta + E\theta_2 - \theta_3 = 0 \qquad \text{......(4.1)}$$

If E_1, E_2, E_3 are the roots of this equation then we must have

$$\left.\begin{aligned} \theta &= E_1 + E_2 + E_3 \\ \theta_2 &= E_1E_2 + E_2E_3 + E_3E_1 \\ \theta_3 &= E_1E_2E_3 \end{aligned}\right\} \qquad \text{......(4.2)}$$

$\theta_1, \theta_2, \theta_3$ are also obtained from the expansion of the determinant in (3.6), and they are given by

$$\theta = E_{11} + E_{22} + E_{33} \qquad \text{......(4.3a)}$$

$$\theta_2 = E_{11}E_{22} + E_{22}E_{33} + E_{33}E_{11} - E_{12}^2 - E_{23}^2 - E_{31}^2$$

$$= \begin{vmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{vmatrix} + \begin{vmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \end{vmatrix} + \begin{vmatrix} E_{33} & E_{31} \\ E_{13} & E_{11} \end{vmatrix} \qquad \text{......(4.3b).}$$

$$\theta_3 = E_{23}^2E_{11} + E_{31}^2E_{22} + E_{12}^2E_{33} - E_{11}E_{22}E_{33} - 2E_{12}E_{23}E_{31}$$

$$= \begin{vmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{vmatrix} \qquad \text{......(4.3c)}$$

Now E_1, E_2, E_3 are the principal strains which have a geometrical meaning independent of the choice of coordinate system. Therefore, θ, θ_2, θ_3 which are given in terms of E_1, E_2, E_3 are also independent of the choice of coordinate system. Hence θ, θ_2, θ_3 as given in (4.3a), (4.3b), (4.3c) are invariants with respect to orthogonal transformations of coordinates. They are respectively called first, second and third strain invariants.

Example 4.1. The strain tensor at a point is given by

$$\left(E_{ij}\right) = \begin{pmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Find principal directions of strain and corresponding direction ratios of principal strains.

Solution : The principal strains are the roots of the equation

$$\begin{vmatrix} a-E & b & 0 \\ b & -a-E & 0 \\ 0 & 0 & -E \end{vmatrix} = 0$$

or, $E\{E^2 - (a^2 + b^2)\} = 0$

Therefore, the roots are $E_1 = 0$, $E_2 = (a^2 + b^2)^{\frac{1}{2}}$, $E_3 = -(a^2 + b^2)^{\frac{1}{2}}$ which are the principal strains. The principal directions are given by

$$\left.\begin{aligned} (a-E)N_1 + bN_2 &= 0 \\ bN_1 - (a+E)N_2 &= 0 \end{aligned}\right\} \qquad \text{......(4.4)}$$
$$-EN_3 = 0$$

For $E_1 = 0$, let $N_i^{(1)} (i = 1, 2, 3)$ be the direction ratios of the principal axis. Then, we have

$$aN_1^{(1)} + bN_2^{(1)} = 0$$
$$bN_1^{(1)} - aN_2^{(1)} = 0$$
$$\text{and } 0.\ N_3^{(1)} = 0$$

The solutions of these equations for $N_i^{(1)}$ are given by

$$N_1^{(1)} = 0,\ N_2^{(1)} = 0,\ N_3^{(1)} = 1$$

For $E_2 = (a^2 + b^2)^{\frac{1}{2}}$, the equations for the direction ratios of the corresponding principal axis are given by

$$\left(a - \sqrt{a^2 + b^2}\right) N_1^{(2)} + bN_2^{(2)} = 0$$
$$bN_1^{(2)} - \left(a + \sqrt{a^2 + b^2}\right) N_2^{(2)} = 0$$
$$\sqrt{a^2 + b^2}\ \ N_3^{(2)} = 0$$

From these equations one can have the solutions as

$$N_1^{(2)} = \frac{a + \sqrt{a^2 + b^2}}{b},\ N_2^{(2)} = 1,\ N_3^{(2)} = 0$$

Similarly, for $E_3 = -(a^2 + b^2)^{\frac{1}{2}}$, we can have

$$N_1^{(3)} = \frac{a - \sqrt{a^2 + b^2}}{b},\ N_2^{(3)} = 1,\ N_3^{(3)} = 0$$

Example 4.2. The displacement in an elastic solid is given by

$$u_1 = a(X_1 + 2X_2 + 3X_3)$$
$$u_2 = a(-2X_1 + X_2)$$
$$u_3 = a(X_1 + 4X_2 + 2X_3)$$

where a is a small quantity. Find dilatation, rotation vector, shear, principal strain and corresponding principal axes.

Solution : We know that

$$E_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right]$$

$$E_{11} = \frac{\partial u_1}{\partial X_1} = a,\ E_{22} = \frac{\partial u_2}{\partial X_2} = a,\ E_{33} = \frac{\partial u_3}{\partial X_3} = 2a$$

$$E_{12} = E_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) = \frac{1}{2}(2a - 2a) = 0$$

$$E_{23} = E_{32} = \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\right) = \frac{1}{2}(0 + 4a) = 2a$$

$$E_{31} = E_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\right) = \frac{1}{2}(3a + a) = 2a$$

$$\therefore \quad \left(E_{ij}\right) = a\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

Dilatation $= \theta = E_{11} + E_{22} + E_{33} = a + a + 2a = 4a$

Rotation vector $\vec{R} = \frac{1}{2}\text{rot } \bar{u} = \frac{1}{2}\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial X_1} & \frac{\partial}{\partial X_2} & \frac{\partial}{\partial X_3} \\ u_1 & u_2 & u_3 \end{vmatrix}$

$$= \frac{1}{2}\left\{i\left(\frac{\partial u_3}{\partial X_2} - \frac{\partial u_2}{\partial X_3}\right) + j\left(\frac{\partial u_1}{\partial X_3} - \frac{\partial u_3}{\partial X_1}\right) + k\left(\frac{\partial u_2}{\partial X_1} - \frac{\partial u_1}{\partial X_2}\right)\right\}$$

$= 2ai + aj - 2ak$

Shear : from (Eij) above we see that

about x_1 axis by an angle $2E_{23} = 4a$,

about x_2 axis by an angle $2E_{31} = 4a$.

Principal strains are the roots of the equation

$$\begin{vmatrix} a - E & 0 & 2a \\ 0 & a - E & 2a \\ 2a & 2a & 2a - E \end{vmatrix} = 0$$

or, $E^3 - 4aE^2 - 3a^2E + 6a^3 = 0$
or, $(E - a)(E^2 - 3aE - 6a^2) = 0$

Therefore, the roots are $E_1 = a$, $E_2 = \dfrac{a\left(3 + \sqrt{33}\right)}{2}$

and $E_3 = \dfrac{a\left(3 - \sqrt{33}\right)}{2}$

Let $N_i^{(1)}$ $(i = 1, 2, 3)$ be the direction ratios of the principal axis corresponding to the strain $E_1 = a$. Then we have the following equations for $N_i^{(1)}$:—

$$(a - E_1) N_1^{(1)} + 2a N_3^{(1)} = 0$$
$$(a - E_1) N_2^{(1)} + 2aN_3^{(1)} = 0$$
$$2aN_1^{(1)} + 2aN_2^{(1)} + (2a - E_1) N_3^{(1)} = 0$$

or, $2a N_3^{(1)} = 0$ and $2aN_1^{(1)} + 2aN_2^{(1)} + aN_3^{(1)} = 0$

$\therefore$ $N_3^{(1)} = 0$ and $N_1^{(1)} = -N_2^{(1)} = 1$ (say)

Therefore, the direction ratios of the principal axis corresponding to $E_1 = a$ are 1, −1, 0. Similarly, for the direction ratios of the other principal axes are $\left(1, 1, \dfrac{8}{\sqrt{33} - 1}\right)$ and $\left(1, 1, -\dfrac{8}{\sqrt{33} + 1}\right)$. .

4.5. COMPATIBILITY RELATIONS

For given strain components E_{ij} as functions of coordinates, the three unknown displacement u_i may be determined from the following equations :

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad ......(5.1)$$

where $u_{ij} = \dfrac{\partial u_i}{\partial X_j}$

Since E_{ij} is symmetric we have six equations in (5.1) for determination of three unknowns. Thus, these equations may not, in geneal, have single-valued solutions u_i for an arbitrarily prescribed values of strain components E_{ij}. That is, one has to put additional restrictions or conditions on the strain compoments to ensure the existence of single-valued displacement solutions. In other words, the strain components must be compatible. We shall now find these compatibility relations for strain compoments E_{ij} as an additional set of partial differential equations to be satisfied by them. For this we try to eliminate displacement u_i from (5.1).

From (5.1), we have

$$E_{ij,k\ell} \equiv \frac{\partial^2 E_{ij}}{\partial X_k \partial X_\ell} = \frac{1}{2}(u_{i,jk\ell} + u_{j,ik\ell}) \quad \ldots\ \ldots(5.2)$$

Again, $E_{k\ell} = \frac{1}{2}\left(u_{k,\ell} + u_{\ell,k}\right)$

$$\therefore \quad E_{k\ell,ij} = \frac{1}{2}\left(u_{k,\ell ij} + u_{\ell,kij}\right) \quad \ldots\ldots(5.3)$$

From (5.2) and (5.3) we have

$$E_{ij,k\ell} + E_{k\ell,ij} = \frac{1}{2}(u_{i,jk\ell} + u_{j,ik\ell} + u_{k,\ell ij} + u_{\ell,kij}) \quad \ldots\ldots(5.4)$$

Interchanging j and k in (5.4) we get

$$E_{ik,j\ell} + E_{j\ell,ik} = \frac{1}{2}\left(u_{i,kj\ell} + u_{k,ij\ell} + u_{j,\ell ik} + u_{\ell,jik}\right) \quad \ldots\ldots(5.5)$$

Now, since $u_{i,jk\ell} = u_{i,kj\ell}$, $u_{\ell,kij} = u_{\ell,jik}$,

$u_{j,ik\ell} = u_{j,\ell ik}$ and $u_{k,\ell ij} = u_{k,ij\ell}$

we have from (5.4) and (5.5)

$$E_{ij,k\ell} + E_{kl,ij} = E_{ik,j\ell} + E_{j\ell,ik}$$

$$\text{or,} \quad E_{ij,k\ell} + E_{k\ell,ij} - E_{ik,j\ell} - E_{j\ell,ik} = 0 \quad \ldots\ldots(5.6)$$

These are Saint Venant's compatibility relations for strain compoments, and are the necessary conditions for the existence of single-valued displacements. The equations in (5.6) form a set of $3^4 = 81$ equations. Out of them only six equations are algebraically independent, that is, none of them can be derived algebraically from other equations, because of the fact that these equations are symmetric with respect to suffixes i, j and k ℓ. These six compatibility equations are

$$\left.\begin{aligned}
&Q_{ij} = 0\\
\text{where } &Q_{11} = \left(E_{22,33} + E_{33,22}\right) - 2E_{23,23}\\
&Q_{22} = \left(E_{33,11} + E_{11,33}\right) - 2E_{31,31}\\
&Q_{33} = \left(E_{11,22} + E_{22,11}\right) - 2E_{12,12}\\
&Q_{23} = Q_{32} = E_{11,23} - \left(-E_{23,1} + E_{31,2} + E_{12,3}\right)_{,1}\\
&Q_{31} = Q_{13} = E_{22,31} - \left(E_{23,1} - E_{31,2} + E_{12,3}\right)_{,2}\\
&Q_{12} = Q_{21} = E_{33,12} - \left(E_{23,1} + E_{31,2} - E_{12,3}\right)_{,3}
\end{aligned}\right\} \quad \ldots\ldots(5.7)$$

Unit : 5 ❑ Analysis of Stress

5.1 BODY AND SURFACE FORCES

Two types of forces, namely (i) external and (ii) internal, act on a continuum body. External forces are exerted by the external agents to the body wheneas internal forces are the bounding forces of interaction among the constituent particles of the continum. The external forces are responsible for deformation of the body, and are (i) body forces (or volume forces) and (ii) surface forces. Surface forces are short range forces and arise from the action of one body on another through the surface of contact if they actually are in contact. For the same continum body one part of it can extert surface force on the other part through the bounding surface of latter. As this force acts on the surface element, the force must be proportional to the area of the surface element. Therefore, these forces are specificed as the forces per unit area.

The body forces are, on the other hand, long-range forces which arise from the action of one body on another while they are at a distance from each other. These forces act equally on all the matter within an element of volume, and one thus specified as force per unit mass. For example, gravitational and magnetic forces are the body forces. The examples of surface force are hydrostatic pressure of liquid or pressure of one solid body on another in contact.

5.2 STRESS VECTOR

Let us consider a deformed continum body every part of which is held in equilibrium under the action of external forces. Also, let us imagine that this body is divided into two parts I and II by a surface Σ within the body (see Fig. 3).

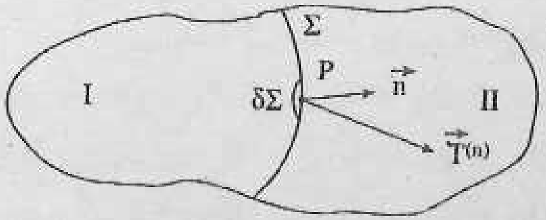


Fig. 3

Now, the part I of the body is in equilibrium under the action of external forces acting on this part and the internal bounding forces of interaction transmitted by the particles of the matter of part II on the particles of matter of part I across the surface Σ. Actually, these internal forces acting at the points of Σ are now external surface forces relative to the part I of the body. Let P (x_1, x_2, x_3) be any point on Σ, and $\delta\Sigma$ be an infinitesimal element of surface Σ of arbitrary size and shape surrounding the point P with outward unit normal $\vec{n}$ at P directed from part I to part II. Then the surface forces distributed over the surface element $\delta\Sigma$ can be resolved into a single restoring force $\delta \vec{F}^{(n)}$ acting at P alomg a definite direction together with a single comple $\delta\vec{G}^{(n)}$ Now, as $\delta\Sigma$ tends to zero always containing the point P within it, the ratio $\frac{\delta \vec{F}^{(n)}}{\delta\Sigma}$ tends to a definite limit $\vec{T}^{(n)}(x_1, x_2, x_3)$ being force per unit area at P, and is called stress vector. Also, the ratio $\frac{\delta \vec{G}^{(n)}}{\delta\Sigma}$ tends to a limit $\vec{M}^{(n)}$ which is called couple stress vector. For most materials this comple stress vector $\vec{M}^{(n)} = 0$ and these continum bodies are called nonpolar. We shall confine ourselves to this type of material. Obviously, the stress vector $\vec{T}^{(n)}$ depends on the positional coordinates (x_1, x_2, x_3) and on the orientation of the particular surface element $\delta\Sigma$ through P, whose exterior unit normal is $\vec{n}$. $\vec{T}^{(n)}$ is taken to be positive if it is directed on the same side of the surface element as in the possitive normal $\vec{n}$ and it tends to restore the material. This vector can be resolved into two component : the normal component along $\vec{n}$, called "normal stress" denoted by $N^{(n)}$ and the tangential component along the tangent to the surface element, called tangential stress denoted by $S^{(n)}$. This tangential stress is also called shearing stress. $N^{(n)}$ is positive if its sense coincides with the sense of outward normal to the surface element at a given point. If $\vec{T}^{(-n)}$ represents the reaction of part I on part II transmitted through the same surface element at (x_1, x_2, x_3), then by Newton's law of action and reaction.

$$\vec{T}^{(-n)}(x_1, x_2, x_3) = -\vec{T}^{(n)}(x_1, x_2, x_3)$$

That is, the stress vector acting on the opposite sides of the same surface element at any given point are equal in magnitude but opposite in sign.

In a continuum body one can associate a stress vector $\vec{T}^{(n)}$ acting on a plane element at any given point with the unit normal vector $\vec{n}$. Since infinite number of planes can be drawn through that point, one has to know all the stress vectors across all the planes for complete specification of the stress at the point. But we can prove that the stress vector at a point on any arbitrary plane surface is a linear function of three stress vectors across any three mutually perpendicular plances through that point.

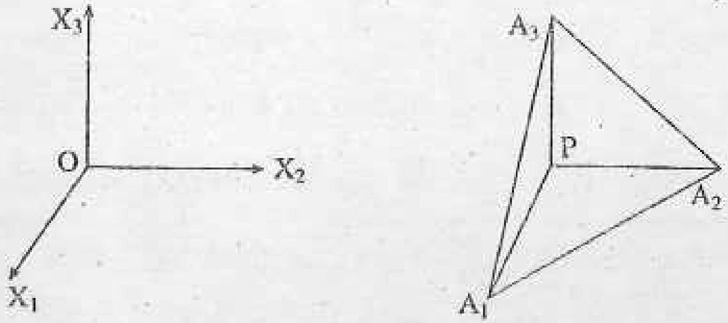


Fig. 4

Let $P(x_1, x_2, x_3)$ be a point of the deformed continuum body. Let a small tetrahedron $PA_1A_2A_3$ having three orthogonal faces PA_2A_3, PA_1A_3, PA_1A_2 parallel to the coordinate plane be an isolated part of the medium. the oblique plane $A_1A_2A_3$ with unit normal $\vec{n}$ is the arbitrary plane which is at a distance d from P. Let us consider the motion of the specific portion of the continuum which occupies the tetrahedron at time t. Let ρ be the density of this portion of continuum. Also, let $\Delta PA_2A_3 = \delta S_1$, $\Delta PA_1A_3 = \delta S_2$, $\Delta PA_1A_2 = \delta S_3$ and $\Delta A_1A_2A_3 = \delta S$.

Then $\delta S_i = n_i\, \delta S \qquad (i = 1, 2, 3) \qquad (2.1)$

Where $\vec{n} = (n_1, n_2, n_3)$

Also, let δV be the volume of the tetrahedron $PA_1A_2A_3$.

$$\text{Then } \delta V = \frac{1}{3} d . \delta S \qquad (2.2)$$

The motion of the tetrahedren is governed by the body forces and the stress vectors across the four boundary planes due to the material outside of it. Now, let

$\vec{\hat{T}}(n)$, $\vec{\hat{T}}(1)$, $\vec{\hat{T}}(2)$, $\vec{\hat{T}}(3)$ be the average stress vectors across the faces $A_1A_2A_3$, PA_2A_3, PA_1A_3, PA_1A_2 respectively. But the stress vectors acting across the plane faces PA_2A_3, PA_1A_3, PA_1A_2 by the material outside of the tetrahedron will be $-\vec{\hat{T}}(1)$, $-\vec{\hat{T}}(2)$, $-\vec{\hat{T}}(3)$ respectively because the exterior normals to these planes are directed oppositely to the positive directions of the axes. Therefore, the equation of motion for the mass of the tetrahedron is

$$\vec{\hat{T}}^{(n)}\delta S - \vec{\hat{T}}^{(1)}\delta S_1 - \vec{\hat{T}}^{(2)}\delta S_2 - \vec{\hat{T}}^{(3)}\delta S_3 + \vec{F}\,\rho\,\delta V = \rho\,\delta V\,\vec{f}$$

where $\vec{F}$ is the body force per unit mass acting on the material inside the tetrahedron having mass $\rho\delta V$. $\vec{f}$ is the acceleration of the material inside the tetrahedron per unit mass. Using (2.1) and (2.2) we have from above equation

$$\vec{\hat{T}}^{(n)}\delta S - \vec{\hat{T}}^{(1)}n_1\delta S - \vec{\hat{T}}^{(2)}n_2\delta S - \vec{\hat{T}}^{(3)}n_3\delta S + \frac{1}{3}\vec{F}\rho d\delta S = \frac{1}{3}\rho d.\,\delta S\,\vec{f}$$

Dividing by δS and taking the limit as $d \to 0$ we have

$$\vec{T}^{(n)} - \vec{T}^{(1)}n_1 - \vec{T}^{(2)}n_2 - \vec{T}^{(3)}n_3 = 0$$

where $\vec{T}^{(n)}$ and $\vec{T}^{(i)}$ $(i = 1, 2, 3)$ are the stress vectors at P transmitted across the aribitrary plane with normal $\vec{n}$ being parallel to the face $A_1A_2A_3$ and across the three mutually perpendicular planes parallel to the coordinate planes respectively. Thus, we have

$$\vec{T}^{(n)} = \vec{T}^{(1)}n_1 + \vec{T}^{(2)}n_2 + \vec{T}^{(3)}n_3 = \vec{T}^{(i)}n_i \qquad (2.3)$$

Thus, the stress vector at a point across the plane with normal $\vec{n}$ is a linear combination of the stress vectors acting on the three orthogonal planes through that point.

5.3. STRESS TENSOR

The side of the plane PA_2A_3 (see Fig. 4) towards which the positive direction of x_1 axis points is called the positive side and other side is called the negative side of the plane. The same convention is followed for the other planes PA_1A_3 and PA_1A_2. Then the stress vector $\vec{T}^{(1)}$ exerts across the plane PA_2A_3 by the material on the positive side of this plane on the material on the negative side of it. This vector can be resolved into the normal stress component along the positive x_1 axis and the shearing stress component. The normal stress is denoted by T_{11}. The shearing stress which acts in the plane PA_2A_3 can further be resolved into two components : along the positive x_2 and x_3 axes and are denoted by T_{12}, T_{13} respectively. Thus,

$$\vec{T}^{(1)} = T_{11}\vec{e}_1 + T_{12}\vec{e}_2 + T_{13}\vec{e}_3 = T_{1i}\vec{e}_i \tag{3.1}$$

where $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are unit vectors along the coordinate axes. Similarly, the stress vectors $\vec{T}^{(2)}, \vec{T}^{(3)}$ acting at P across the planes PA_3A_1 and PA_1A_2 respectively can be resolved. We have, therefore, the relations

$$\vec{T}^{(2)} = T_{21}\vec{e}_1 + T_{22}\vec{e}_2 + T_{23}\vec{e}_3 = T_{2i}\vec{e}_i \tag{3.2}$$

and $$\vec{T}^{(3)} = T_{31}\vec{e}_1 + T_{32}\vec{e}_2 + T_{33}\vec{e}_3 = T_{3i}\vec{e}_i \tag{3.3}$$

We can write, from (3.1), (3.2) and (3.3),

$$\vec{T}^{(i)} = T_{ij}\vec{e}_j \tag{3.4}$$

Thus, T_{ij} is the j th component of the stress vector $\vec{T}^{(i)}$ at P acting across a plane normal to x_i axis. Now, $\vec{T}^{(n)}$ is the stress vector at P across a plane normal to $\vec{n} = (n_1, n_2, n_3)$.

Then, $\vec{T}^{(n)} = \vec{T}^{(i)} n_i$ (from (2.3))

$$= T_{ij}\vec{e}_j n_i = \vec{e}_j\left(T_{ij}n_i\right)$$

$$= \vec{e}_i\left(T_{ji}n_j\right) \text{ (interchanging dummy suffixes)}$$

or, $T_i^{(n)}\vec{e}_i = \vec{e}_i\left(T_{ji}\ n_j\right)$ where $T_i^{(n)}$ are components of $\vec{T}^{(n)}$

$$\therefore\ T_i^{(n)} = T_{ji}\ n_j.$$

Thus, the stress vector at P across any arbitrary plane normal to $\vec{n}$ is a linear combination of nine stress component T_{ji} acting across three mutually perpendicular planes at the point parallel to the coordinate planes. That is, the stress at any point of the continum is completely specified by nine components T_{ij}. Now, n_i is an arbitrany vector, and also $T_{ji}\, n_j$ represents the compenents of a vector $\vec{T}^{(n)}$. Then from the quotient law of tensor it follows that T_{ij} form a tensor of second order, and is called the stress tensor at P. The components T_{11}, T_{22}, T_{33} are called normal stresses, and T_{12}, T_{23}, T_{31}, T_{21}, T_{32}, T_{13} are called shearing stresses. The normal stress on any arbitrary plane normal to $\vec{n}$ is given by

$$N^{(n)} = T_i^{(n)} n_i = T_{ji} n_j n_i = T_{ij} n_i n_j \tag{3.6}$$

If $S^{(n)}$ be the magnitude of the shearing stress, then we have

$$\left(T_1^{(n)}\right)^2 + \left(T_2^{(n)}\right)^2 + \left(T_3^{(n)}\right)^2 = \left(N^{(n)}\right)^2 + \left(S^{(n)}\right)^2$$

or, $$\left(S^{(n)}\right)^2 = \left(T_1^{(n)}\right)^2 + \left(T_2^{(n)}\right)^2 + \left(T_3^{(n)}\right)^2 - \left(N^{(n)}\right)^2$$

$$= \left(T_{j1}\, n_j\right)^2 + \left(T_{j2}\, n_j\right)^2 + \left(T_{j3}\, n_j\right)^2 - \left(T_{ij}\, n_i\, n_j\right)^2 \tag{3.7}$$

5.4 STRESS EQUATIONS OF EQUILIBRIUM AND MOTION

Let us use Lagrangian method. Consider a specific portion of the undeformed continuum occupying initially (at $t = 0$) an arbitrary volume V_0. Let $\rho_0 = \rho(X_1, X_2, X_3)$ be the density at the point $P_0(X_1, X_2, X_3)$. Therefore, the total mass of the continuum in V_0 at $t = 0$ is given by $\iiint\limits_{V_0} \rho_0 dV_0$ where dV_0 is the element of volume at P_0.

At the subsequent time $t > 0$, the material in V_0 moves in such manner that they will be in some other volume V in the deformed state. Let the particle at P_0 at $t = 0$ occupy the position $P(x_1, x_2, x_3)$ in V at t. Then the equations charaterizing the motion in Lagrangian method are

$$\left.\begin{aligned} x_i &= x_i\left(X_1, X_2, X_3, t\right) \\ \text{and}\quad \rho &= \rho\left(X_1, X_2, X_3, t\right) \end{aligned}\right\} \qquad \ldots\ldots(4.1)$$

where ρ is the density of the continuum at P.

Therefore, the total matter in V at t

$$= \iiint_V \rho dV$$

By the principle of conservation of mass, we must have

$$\iiint_{V_0} \rho_0 dV_0 = \iiint_V \rho dV \qquad \text{......(4.2)}$$

Since $\quad dV = J dV_0 = \dfrac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} dV_0 \qquad \text{......(4.3)}$

where
$$\frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} = \begin{vmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_3}{\partial X_1} \\ \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_3}{\partial X_2} \\ \dfrac{\partial x_1}{\partial X_3} & \dfrac{\partial x_2}{\partial X_3} & \dfrac{\partial x_3}{\partial X_3} \end{vmatrix}$$

we have $\displaystyle\iiint_{V_0} \rho_0 dV_0 = \iiint_{V_0} \rho \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} dV_0$

or, $$\iiint_{V_0} \left(\rho_0 - \rho \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} \right) dV_0 = 0$$

Since the volume V_0 is arbitrary, it follows that the integrand must vanish at every point of the continuum, that is,

$$\rho_0 = \rho \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} \qquad \text{......(4.4)}$$

This is the equation of continuity in Lagrangian method.

We can find the equation of continuity in Eulerian method. Here, the principle of conservation of mass is expressed as the rate of increase of mass of the continuum within any fixed closed surface being equated to the rate of net mass flow across the boundary surface within it. Let S be the any fixed closed surface enclosing a volume V lying entirely in a region through which continum moves. Let $P(x_1, x_2, x_3)$ be any point in it and ρ be the density at P at time t. So that

$$\rho = \rho\ (x_1, x_2, x_3, t)$$

The total mass in V is $\iiint_V \rho dv$ Therefore, the net local rate of the increase of this mass in V is $\frac{\partial}{\partial t}\left(\iiint_V \rho dV\right) = \iiint_V \frac{\partial \rho}{\partial t} dV$, since V is a fixed region of space, and the coordinates x_1, x_2, x_3 are independent of t. Now, let $\vec{v}$ be the velocity of the particle at a point Q on the surface S. Let $\vec{n}$ be the outward drawn normal to the surface element at Q. Therefore, $\rho\, \vec{n} . \vec{v} dS$ is the mass of the continuum leaving volume V with the flow across dS per unit time. Thus, the mass of the continuum entering into V through dS per unit time is $-\rho\, \vec{n} . \vec{v}\ dS$. Then the rate of net mass flow across the total boundary sunface S is $-\iiint_S \rho\, \vec{n} . \vec{v} dS$. By the principle of conservation of mass we have

$$\iiint_V \frac{\partial \rho}{\partial t} dV = -\iint_S \rho\, \vec{n} . \vec{v}\ dS \qquad \text{......(4.5)}$$

Now, by Gauss's divergence theorem we have

$$\iint_S \rho\, \vec{n} . \vec{v}\ ds = \iiint_V \text{div.}\left(\rho\, \vec{v}\right) dV$$

Therefore, from (4.5) we get

$$\iiint_V \frac{\partial \rho}{\partial t} = -\iiint_V \text{div}\left(\rho\, \vec{v}\right) dV$$

$$\text{or,} \quad \iiint_V \left(\frac{\partial \rho}{\partial t} + \text{div}\left(\rho\, \vec{v}\right)\right) dV = 0$$

Since V is an arbitrary volume, the integrand must vanish at every point of the continuum. Therefore,

$$\frac{\partial \rho}{\partial t} + \text{div}\left(\rho \vec{v}\right) = 0 \qquad \text{......(4.6)}$$

This is the equation of continuity in Eulerian method.

Exercise 4.1. Prove that the two forms of equations of continuity one equivalent.

Proof, Let us start from the equation of continuity in Lagrangian method, which is $\rho J = \rho_0$ where the Jacobian $J = \dfrac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)}$. Taking total derivative of this equation with respect to t, we get

$$\frac{d}{dt}(\rho J) = 0$$

or, $$\frac{d\rho}{dt} J + \rho \frac{dJ}{dt} = 0 \qquad \text{......(4.7)}$$

Now, $$\frac{dJ}{dt} = \frac{d}{dt}\left\{\frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)}\right\} = \frac{d}{dt}\begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_2}{\partial X_1} & \frac{\partial x_3}{\partial X_1} \\ \frac{\partial x_1}{\partial X_2} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_3}{\partial X_2} \\ \frac{\partial x_1}{\partial X_3} & \frac{\partial x_2}{\partial X_3} & \frac{\partial x_3}{\partial X_3} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{d}{dt}\left(\frac{\partial x_1}{\partial X_1}\right) & \frac{\partial x_2}{\partial X_1} & \frac{\partial x_3}{\partial X_1} \\ \frac{d}{dt}\left(\frac{\partial x_1}{\partial X_2}\right) & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_3}{\partial X_2} \\ \frac{d}{dt}\left(\frac{\partial x_1}{\partial X_3}\right) & \frac{\partial x_2}{\partial X_3} & \frac{\partial x_3}{\partial X_3} \end{vmatrix} + \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{d}{dt}\left(\frac{\partial x_2}{\partial X_1}\right) & \frac{\partial x_3}{\partial X_1} \\ \frac{\partial x_1}{\partial X_2} & \frac{d}{dt}\left(\frac{\partial x_2}{\partial X2}\right) & \frac{\partial x_3}{\partial X_2} \\ \frac{\partial x_1}{\partial X_3} & \frac{d}{dt}\left(\frac{\partial x_2}{\partial X_3}\right) & \frac{\partial x_3}{\partial X_3} \end{vmatrix}$$

$$+ \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_2}{\partial X_1} & \frac{d}{dt}\left(\frac{\partial x_3}{\partial X_1}\right) \\ \frac{\partial x_1}{\partial X_2} & \frac{\partial x_2}{\partial X_2} & \frac{d}{dt}\left(\frac{\partial x_3}{\partial X_2}\right) \\ \frac{\partial x_1}{\partial X_3} & \frac{\partial x_2}{\partial X_3} & \frac{d}{dt}\left(\frac{\partial x_3}{\partial X_3}\right) \end{vmatrix} \qquad \text{......(4.8)}$$

1st determinant on the R.H.S. of (4.8)

$$= \begin{vmatrix} \frac{\partial}{\partial X_1}\left(\frac{dx_1}{dt}\right) & \frac{\partial x_2}{\partial X_1} & \frac{\partial x_3}{\partial X_1} \\ \frac{\partial}{\partial X_2}\left(\frac{dx_1}{dt}\right) & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_3}{\partial X_2} \\ \frac{\partial}{\partial X_3}\left(\frac{dx_1}{dt}\right) & \frac{\partial x_2}{\partial X_3} & \frac{\partial x_3}{\partial X_3} \end{vmatrix} = \begin{vmatrix} \frac{\partial v_1}{\partial X_1} & \frac{\partial x_2}{\partial X_1} & \frac{\partial x_3}{\partial X_1} \\ \frac{\partial v_1}{\partial X_2} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_3}{\partial X_2} \\ \frac{\partial v_1}{\partial X_3} & \frac{\partial x_2}{\partial X_3} & \frac{\partial x_3}{\partial X_3} \end{vmatrix}$$

where (v_1, v_2, v_3) are velocity components of the particle instantaneously occuping the point (x_1, x_2, x_3). Then the above term

$$= \frac{\partial(v_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} = \frac{\partial(v_1, x_2, x_3)}{\partial(x_1, x_2, x_3)} \cdot \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)}$$

$$= J \begin{vmatrix} \frac{\partial v_1}{\partial x_1} & 0 & 0 \\ \frac{\partial v_1}{\partial x_2} & 1 & 0 \\ \frac{\partial v_1}{\partial x_3} & 0 & 1 \end{vmatrix} = J \frac{\partial v_1}{\partial x_1}$$

Similarly, the second and third terms of the R.H.S. of (4.8) are respectively $J\frac{\partial v_2}{\partial x_2}$ and $J\frac{\partial v_3}{\partial x_3}$. Therefore, from (4.8)

$$\frac{dJ}{dt} = J\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right) = J \text{ div } \vec{v} \quad \ldots\ldots(4.9)$$

Substituting (4.9) into (4.7) we have

$$\frac{d\rho}{dt} J + \rho J \text{ div } \vec{v} = 0$$

or, $\quad \frac{d\rho}{dt} + \rho \text{ div } \vec{v} = 0$

or, $\quad \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{dx_i}\frac{dx_i}{dt} + \rho \text{ div } \vec{v} = 0$

or, $\quad \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_i} v_i + \rho \text{ div } \vec{v} = 0$

or, $\frac{\partial\rho}{\partial t} + \text{div}(\rho\vec{v}) = 0$ [Since $\text{div}(\rho\vec{v}) = \vec{v}.\text{grad}\,\rho + \rho\,\text{div}\,\vec{v}$

$= v_i \frac{\partial\rho}{\partial x_i} + \rho\,\text{div}\,\vec{v}$]

which is the equation of continuity in Eulerian method.

Now we use the principle of balance of linear momentum to derive the equation of motion. This principle states that the time rate of change of total linear momentum of a specific portion of the continuum is equal to the resultant external force acting on this portion.

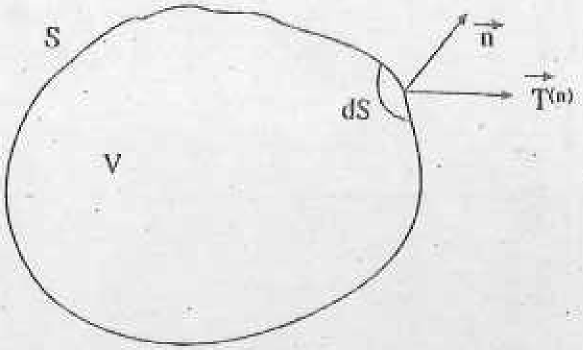


Fig. 5

Let the specific portion of the deformed continuum occupy an orbitary volume V bounded by the closed surface S at time t. Let $\rho(x_1, x_2, x_3, t)$ and $\vec{v}(x_1, x_2, x_3)$ be the density field and velocity field of the continuum. The linear momentum $\vec{P}(t)$ of this specific portion of the continuum within V at time t is given by

$$\vec{P}(t) = \iiint_V \rho\vec{v}\, dV \qquad \text{......(4.10)}$$

Let $\vec{F}(x_1, x_2, x_3, t)$ be the field of body force per unit mass, and $\vec{T}^{(n)}(x_1, x_2, x_3, t)$ be the field of surface force per unit area of the surface. This surface force is the stress vector acting across the surface element with outward unit normal vector $\vec{n}$ exerted by the surrounding material of the portion V (Fig. 5). The resultant external force $\vec{R}(t)$ acting on the portion of the continuum in V at time t is given by

$$\vec{R}(t) = \iiint_V \rho\,\vec{F}\, dV + \iint_S T^{(\vec{n})}\, dS \qquad \text{......(4.11)}$$

By the principle of balance of linear momentum we have

$$\frac{d}{dt}\left\{\iiint_V \rho\vec{v}\,dV\right\} = \iiint_V \rho\,\vec{F}\,dV + \iint_S \vec{T}^{(n)}dS$$

or, $$\iiint_V \rho\frac{d\vec{v}}{dt}dV + \iiint_V \vec{v}\frac{d}{dt}(\rho dV) = \iiint_V \rho\,\vec{F}\,dV + \iint_S \vec{T}^{(n)}dS \qquad \ldots\ldots(4.12)$$

Now, $$\frac{d}{dt}(\rho dV) = \frac{d\rho}{dt}dV + \rho\frac{d}{dt}(dV) = \frac{d\rho}{dt}dV + \rho\frac{d}{dt}(JdV_0)$$

$$= \frac{d\rho}{dt}dV + \rho dV_0\frac{dJ}{dt} = \frac{d\rho}{dt}dV + \rho dV_0 J\ \operatorname{div}\vec{v} \text{ (using (4.9))}$$

$$= \frac{d\rho}{dt}dV + \rho\operatorname{div}\vec{v}\,dV$$

$$= \left(\frac{d\rho}{dt} + \rho\operatorname{div}\vec{v}\right)dV$$

$$= \left(\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\vec{v})\right)dV = 0 \qquad \ldots\ldots(4.13)$$

(by using the equation of continuity (4.6)

Therefore from (4.12) we have

$$\iiint_V \rho\frac{d\vec{v}}{dt}dV = \iiint_V \rho\,\vec{F}\,dV + \iint_S \vec{T}^{(n)}dS$$

or, $$\iiint_V \rho\frac{dv_i}{dt}dV = \iiint_V \rho F_i\,dV + \iint_S T_i^{(n)}dS \qquad \ldots\ldots(4.14)$$

Again, we use Gauss's divergence theorem to obtain

$$\iint_S T_i^{(n)}\,dS = \iint_S T_{ji}n_j\,dS = \iiint_V T_{ji,j}\,dV \qquad \ldots\ldots(4.15)$$

$\therefore$ $$\iiint_V \rho\frac{dv_i}{dt}dV = \iiint_V \rho F_i\,dV + \iiint_V T_{ji,j}\,dV$$

or, $$\iiint_V \left(\rho F_i + T_{ji,j} - \rho\dot{v}_i\right)dV = 0$$

Since V is arbitrary volume, we have at every point of the continuum the following equation of motion :

$$\rho F_i - T_{ji,j} - \rho \dot{v}_i = 0 \qquad \text{......(4.16)}$$

If $u_i(x_1, x_2, x_3, t)$ is the displacement field, then $v_i = \dot{u}_i$ and the above equation of motion takes the following form :

$$\rho F_i + T_{ji,j} - \rho \ddot{u}_i = 0 \qquad \text{......(4.17)}$$

If the continuum is in static equilibrium, the acceleration $\dot{v}_i$ vanishes, and we have the following equation of equilibrium :

$$\rho F_i + T_{ji,j} = 0 \qquad \text{......(4.18)}$$

5.5. SYMMETRY OF STRESS TENSOR

We use the principle of balance of angular momentum to establish the symmetry of stress tensor. This principle states that the time rate of change of total angular momentum of a specific portion of the continuum about an arbitrary point is equal to the resultant moment about the same point of the external force acting on this portion.

Let $\vec{H}(t)$ be the total angular mementuum about the origin of coordinates of the portion of the continuum in V at time t (see Fig. 5). Then

$$\vec{H}(t) = \iiint_V (\vec{r} \times \vec{v}) \rho \, dV \qquad \text{......(5.1)}$$

or its ith component

$$H_i(t) = \iiint_V \epsilon_{ijk} \, x_j \, \rho v_k \, dV \qquad \text{......(5.2)}$$

where,

$$\left.\begin{aligned} \epsilon_{ijk} &= 0 \text{ if any two of i, j, k are equal} \\ &= 1 \text{ if i, j, k are even permutation of 1, 2, 3} \\ &= -1 \text{ if i, j, k are odd permutation of 1, 2, 3} \end{aligned}\right\} \qquad \text{......(5.3)}$$

If $M_i(t)$ is ith component of the moment of external force acting on the portion in V about the origin at time t, then

$$M_i(t) = \iiint_V \epsilon_{ijk}\ x_j\,\rho F_k dV + \iint_S \epsilon_{ijk}\ x_j\ T_k^{(n)} dS \quad \text{......(5.3)}$$

Therefore, by the principle of balance of angular momentum we have

$$\frac{d}{dt}\left(\iiint_V \epsilon_{ijk}\ x_j\,\rho v_k\ dV\right) = \iiint_V \epsilon_{ijk}\ x_j\,\rho F_k\ dV + \iint_S \epsilon_{ijk}\ x_j\ T_k^{(n)} dS \quad \text{...(5.4)}$$

$$\text{L.H.S. of (5.4)} = \iiint_V \epsilon_{ijk}\ \frac{d}{dt}\left(x_j\,\rho v_k dV\right)$$

$$= \iiint_V \epsilon_{ijk}\ x_j v_k \frac{d}{dt}(\rho dV) + \iiint_V \epsilon_{ijk}\ \frac{d}{dt}\left(x_j v_k\right)\rho\ dV$$

$$= \iiint_V \epsilon_{ijk}\left(\dot{x}_j v_k + x_j \dot{v}_k\right)\rho dV \quad \left(\text{since } \frac{d}{dt}(\rho dV) = 0 \text{ by (4.13)}\right)$$

Now, $x_i = X_i + u_i$, u_i being displacement field, and then $\dot{x}_i = \dot{u}_i = v_i$. With this, the L.H.S. of (5.4) becomes

$$\iiint_V \epsilon_{ijk}\ v_j\,v_k\,\rho dV + \iiint_V \epsilon_{ijk}\ x_j\ \dot{v}_k\ \rho dV$$

Now, $\epsilon_{ijk}\ v_j\,v_k = \epsilon_{ikj}\ v_k\ v_j$ (interchanging dummy suffixes)

$$= -\ \epsilon_{ijk}\ v_k\,v_j \quad (\text{since } \epsilon_{ijk} = -\ \epsilon_{ikj})$$

$\therefore \quad 2\,\epsilon_{ijk}\ v_j\ v_k = 0$

Consequently, the L.H.S. of (5.4) $= \iiint_V \epsilon_{ijk}\ x_j\ \dot{v}_k\ \rho\ dV$ ......(5.5)

Again, $\iint_S \epsilon_{ijk}\ x_j\ T_k^{(n)} dS = \iint_S \epsilon_{ijk}\ x_j\ T_{\ell k}\ n_\ell\ dS$

$$= \iiint_V \operatorname{div}\left(\epsilon_{ijk}\ x_j\ T_{\ell k}\right) dV \quad (\text{using Gauss's theorem})$$

$$= \iiint_V \epsilon_{ijk}\ \left(x_j\ T_{\ell k}\right)_{,\ell}\ dV \quad (\epsilon_{ijk} \text{ are constants})$$

$$= \iiint_V \epsilon_{ijk} \left(x_{j,\ell} T_{\ell k} + x_j T_{\ell k,\ell} \right) dV$$

$$= \iiint_V \epsilon_{ijk} \left(\delta_{j\ell} T_{\ell k} + x_j T_{\ell k,\ell} \right) dV$$

$$= \iiint_V \epsilon_{ijk} \left(T_{jk} + x_j T_{\ell k,\ell} \right) dV \qquad \text{......(5.6)}$$

Using (5.5) and (5.6) we have from (5.4)

$$\iiint_V \epsilon_{ijk}\ x_j\ \dot{v}_k\ \rho dV = \iiint_V \epsilon_{ijk}\ x_j\ \rho F_k\ dV + \iiint_V \epsilon_{ijk} \left(T_{jk} + x_j T_{\ell k,\ell} \right) dV$$

or, $$\iiint_V \epsilon_{ijk}\ T_{jk}\ dV = \iiint_V \epsilon_{ijk} \left(\dot{v}_k \rho - \rho F_k - T_{\ell k,\ell} \right) x_j\ dV$$

$$= 0 \ \text{(by using the equation of motion (4.16))}$$

$\therefore\ \epsilon_{ijk} T_{jk} = 0$ at every point of the continuum since the volume V is arbitrary

Let $i = 1$, then we have from above

$$\epsilon_{1jk} T_{jk} = 0 \text{ or, } \epsilon_{123} T_{23} + \epsilon_{132} T_{32} = 0$$

or, $\quad T_{23} - T_{32} = 0 \quad$ or, $\ T_{23} = T_{32}$

Similarly taking $i = 2$, we have $T_{31} = T_{13}$, and taking $i = 3$, we have $T_{12} = T_{21}$

Thus, the stress tensor is symmetric, and the stress at any point of the continuum is completely specified by the six components (instead of nine components) of the stress tenser.

With this symmetry of stress tensor the equation of motion (4.17) can be written as

$$\rho F_i + T_{ij,j} = \rho \ddot{u}_i = \rho \dot{v}_i \qquad \text{......(5.7)}$$

5.6 STRESS QUADRIC

With the origin at $P(x_1, x_2, x_3)$ in the deformed continuum let us, introduce a local coordinate system $P\xi_1, P\xi_2, P\xi_3$ parallel to OX_1, OX_2, OX_3 respectively. Then for a given set of stress tensor T_{ij} the quadric surface with its centre at P given by

$$T_{ij}\, \xi_i\, \xi_j = 1 \qquad \ldots\ldots(6.1)$$

is called "stress quadric"

Exercise 6.1. Prove that the normal stress across any plane through the centre of stress quadric is equal to the inverse of the square of the central radius vector of the quadric normal to the plane.

Proof : Let Q be a point on the stress quadric (6.1)

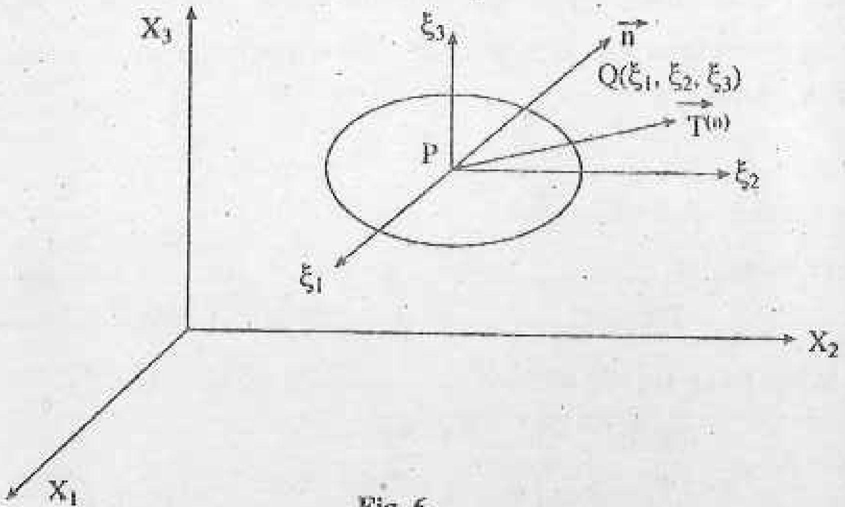


Fig. 6

Let us consider a plane element through P normal to PQ (Fig. 6). Let $\bar{n} = (n_1, n_2, n_3)$, n_1, n_2, n_3 being direction cosines of PQ, and r is the length of PQ. Then the stress vector across that plane element is given by

$$T_i^{(n)} = T_{ij}\, n_j$$

The normal stress at P across the plane normal to $\bar{n}$ is given by

$$N^{(n)} = T_i^{(n)}\, n_i = T_{ij}\, n_i\, n_j$$

Also, also coordinates (ξ_1, ξ_2, ξ_3) of Q are given as

$$\xi_i = rn_i \quad \text{or,} \quad n_i = \frac{\xi_i}{r}$$

Therefore, $$N^{(n)} = \frac{T_{ij}\xi_i\xi_j}{r^2} = \frac{1}{r^2} \quad \text{(proved)}$$

(since Q lies on the stress quadric)

5.7 PRINCIPAL STRESS

When the stress vector is along the direction perpendicular to the element of plane on which it acts, it is called principal stress. The element of plane on which principal stress is acting is called principal plane, and the direction of principal stress is known as principal axis of stress or principal direction of stress.

Let $P(x_1, x_2, x_3)$ be any point in the continuum. Let $\bar{n} = (n_1, n_2, n_3)$ be the unit normal vector to an element of plane through the point P. Also let $\vec{T}^{(n)}$ be the stress vector acting across this element of plane. If $\vec{T}^{(n)}$ is a principal stress, it must be along the normal $\bar{n}$, that is

$$T_i^{(n)} = Tn_i \qquad \text{......(7.1)}$$

where T is the magnitude of $\vec{T}^{(n)}$

Also,we know that

$$T_i^{(n)} = T_{ij}\, n_j \qquad \text{......(7.2)}$$

Then from (7.1) and (7.2) we have

$$T_{ij}\, n_j = Tn_i = T\,\delta_{ij}\, n_j$$

or, $$\left(T_{ij} - T\delta_{ij}\right) n_j = 0 \qquad \text{......(7.3)}$$

or, expanding in detail

$$\left.\begin{aligned} (T_{11} - T)n_1 + T_{12}n_2 + T_{13}n_3 &= 0 \\ T_{21}n_1 + (T_{22} - T)n_2 + T_{23}n_3 &= 0 \\ T_{31}n_1 + T_{32}n_2 + (T_{33} - T)n_3 &= 0 \end{aligned}\right\} \text{......(7.4)}$$

Also we have $n_i\, n_i = 1$ ......(7.5)

The equations (7.4) for n_1, n_2, n_3 have the trivial solution $n_1 = n_2 = n_3 = 0$, which is not compatible with the equation (7.5). For existence of nontrivial solution of (7.4) we must have

$$\begin{vmatrix} T_{11} - T & T_{12} & T_{13} \\ T_{21} & T_{22} - T & T_{23} \\ T_{31} & T_{32} & T_{33} - T \end{vmatrix} = 0 \quad \text{......(7.6)}$$

which is the characteristic equation for the determination of principal stresses. The cubic equation in T possesses three roots T_1, T_2, T_3, the principal stresses. Substituting for each of the principal stresses in (7.4) and using (7.5), one can find the three sets of solutions of n_1, n_2, n_3 giving rise to the direction cosines for three principal axes of stresses.

Exercise 7.1. Prove that roots of the characteristic equation determining principal stresses are real. Prove also that the directions corresponding to the principal stresses T_1, T_2, T_3 are mutually perpendicular.

Hints : Proofs are same as that in the case of principal strains.

Now the cubic eqution (7.6) can be written as

$$T^3 - \Theta T^2 + \Theta_2 T - \Theta_3 = 0 \qquad \ldots\ldots(7.7)$$

where $\Theta = T_{11} + T_{22} + T_{33}$

$$\left.\begin{aligned} \Theta_2 &= \begin{vmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{23} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{33} & T_{31} \\ T_{31} & T_{11} \end{vmatrix} \\ \Theta_3 &= \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} \end{aligned}\right\} \qquad \ldots\ldots(7.8)$$

Also, the roots T_1, T_2, T_3 of the cubic equation (7.7) are related to the coefficient Θ, Θ_2, Θ_3 by the following relations

$$\left.\begin{aligned} \Theta &= T_1 + T_2 + T_3 \\ \Theta_2 &= T_1T_2 + T_2T_3 + T_3T_1 \\ \Theta_3 &= T_1T_2T_3 \end{aligned}\right\} \qquad \ldots\ldots(7.9)$$

The principal stresses T_1, T_2, T_3 at a point do not depend on the choice of coordinate system, and therefore, Θ, Θ_2, Θ_3 given in (7.8) are invariant under an orthogonal transformation of coordinate axes. These are called, respectively, the first, second and third invariants.

Example 7.1. The stress tensor at P is given by

$$\left(T_{ij}\right) = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

Determine principal stresses and principal directions.

Ans. The characteristic equation is

$$\begin{vmatrix} 3-T & 1 & 1 \\ 1 & -T & 2 \\ 1 & 2 & -T \end{vmatrix} = 0$$

or, $T^3 - 3T^2 - 6T + 8 = 0$

or, $(T-1)(T^2 - 2T - 8) = 0$

$\therefore \quad T = 1, T = \dfrac{2 \pm \sqrt{4+32}}{2} = 1 \pm 3 = 4, -2$

$\therefore \quad T_1 = 1, T_2 = 4, T_3 = -2$

For $T_1 = 1$ the principal direction is given by the solution of the following equations.

$$(3-1)n_1 + n_2 + n_3 = 0 \quad \text{or,} \quad 2n_1 + n_2 + n_3 = 0$$

$$n_1 - n_2 + 2n_3 = 0, \quad n_1 + 2n_2 - n_3 = 0$$

$$n_1^2 + n_2^2 + n_3^2 = 1$$

$\therefore \quad 3n_1 + 3n_3 = 0 \quad \text{or,} \quad n_1 = -n_3, \quad \therefore \quad 2n_1 + n_2 - n_1 = 0, \quad \text{or,} \quad n_1 = -n_2$

Also, we have $n_1^2 + n_1^2 + n_1^2 = 1, \ \therefore \ 3n_1^2 = 1, \ n_1 = \dfrac{1}{\sqrt{3}}$

$\therefore \quad n_1 = \dfrac{1}{\sqrt{3}}, \ n_2 = -\dfrac{1}{\sqrt{3}}, \ n_3 = -\dfrac{1}{\sqrt{3}}$

Similarly, we can find the directions of other principal stresses as $\dfrac{2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}$ and $0, \dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}$

Example 7.2. The state of stress at a point is given by

$$(T_{ij}) = \begin{pmatrix} T & aT & bT \\ aT & T & cT \\ bT & cT & T \end{pmatrix}$$

where a, b, c are constants and T is some stress value. Determine the constants a, b, c so that the strress vector on a plane normal to $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ vanishes.

Ans. $\vec{T}_i^{(n)} = T_{ji}n_j = T_{ij}n_j, \quad \vec{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$$T_1^{(n)} = T_{1i}n_i = (T + aT + bT)\frac{1}{\sqrt{3}} = 0$$

$$T_2^{(n)} = T_{2i}n_i = (aT + T + cT)\frac{1}{\sqrt{3}} = 0$$

$$T_3^{(n)} = T_{3i}n_i = (bT + cT + T)\frac{1}{\sqrt{3}} = 0$$

if the stress vector $\vec{T}^{(n)}$ on a plane normal to $\vec{n}$ vanishes.

$$\therefore \quad 1 + a + b = 0$$

$$a + 1 + c = 0 \qquad (T \neq 0)$$

$$b + c + 1 = 0$$

Adding $\quad 2(a + b + c) + 3 = 0$

or, $\quad a + b + c = -\frac{3}{2}$

Sicne $a + b + 1 = 0$ we have $c = -\frac{1}{2}$. Similarly, $a = b = -\frac{1}{2}$.

Example 7.3. Determine the Cauchy's stress quadric at P for a state of stress

$$\left(T_{ij}\right) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

where a, b, c are all of same sign.

Ans. Cauchy's stress quadric is $\quad T_{ij}\,\xi_i\,\xi_j = 1$

or, $\quad T_{11}\xi_1^2 + T_{22}\xi_2^2 + T_{33}\xi_3^2 = 1$

or, $\quad a\xi_1^2 + b\xi_2^2 + c\xi_3^2 = 1$

or, $\quad \frac{\xi_1^2}{1/a} + \frac{\xi_2^2}{1/b} + \frac{\xi_3^2}{1/c} = 1$

which is an ellipsoid

Example 7.4. The principal stresses at a point are $T_1 = 1, T_2 = -1,$

$T_3 = 3$. If stress at a point is given by

$$\left(T_{ij}\right) = \begin{pmatrix} T_{11} & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & T_{33} \end{pmatrix}$$

Find the value of T_{11} and T_{33}.

Ans. Stress invariants are

$$\Theta = T_{11} + T_{22} + T_{33} = T_1 + T_2 + T_3 = 1 - 1 + 3 = 3$$

$$\Theta_2 = \begin{vmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{23} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{33} & T_{31} \\ T_{31} & T_{11} \end{vmatrix} = T_1T_2 + T_2T_3 + T_3T_1$$

$$= -1 - 3 + 3 = -1$$

$$\Theta_3 = \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} = T_1T_2T_3 = -3$$

or, $T_{11} + 1 + T_{33} = 3$ or, $T_{11} + T_{33} = 2$ ......(a)

$$\Theta_2 = \begin{vmatrix} T_{11} & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & T_{33} \end{vmatrix} + \begin{vmatrix} T_{33} & 0 \\ 0 & T_{11} \end{vmatrix} = -1$$

or, $T_{11} + T_{33} - 4 + T_{33}T_{11} = -1$

or, $T_{11} + T_{33} + T_{11}T_{33} = 3$ ......(b)

$$\Theta_3 = \begin{vmatrix} T_{11} & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & T_{33} \end{vmatrix} = -3 \quad \text{or,} \quad T_{11}(T_{33} - 4) = -3$$

or, $T_{11}T_{33} - 4T_{11} = -3$ ......(c)

From (a) and (b), $2 + T_{11}T_{33} = 3$ or, $T_{11}T_{33} = 1$

Then from (c), $1 - 4T_{11} = -3$ or, $4T_{11} = 4$, $\therefore$ $T_{11} = 1$

and $T_{33} = 1$

Example 7.5. Given the following stress distribution

$$\left(T_{ij}\right) = \begin{pmatrix} x_2 & -x_3 & 0 \\ -x_3 & 0 & -x_2 \\ 0 & -x_2 & T \end{pmatrix}$$

find T such that stress distribution is in equilibrium with the body force $\vec{F} = -g\vec{e}_3$.

Ans. The equation of equilibrium are given by

$$\rho F_i + T_{ij,j} = 0$$

In this case $\vec{F} = -g\vec{e}_3$

$$\therefore \quad F_1 = 0,\ F_2 = 0 \text{ and } F_3 = -g$$

The equations of equilibrium are, then,

$$T_{1j,j} = 0,\ T_{2j,j} = 0 \text{ and } -\rho g + T_{3j,j} = 0$$

or, these equations are

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} = 0$$

$$\frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} = 0$$

$$-\rho g + \frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} = 0$$

Obviously, for the given stress distribution the first two equations are identically satisfied. From the third equation we have

$$-\rho g + 0 + \frac{\partial(-x_2)}{\partial x_2} + \frac{\partial T}{\partial x_3} = 0$$

$$\text{or,} \quad \frac{\partial T}{\partial x_3} = 1 + \rho g$$

Integrating we have, $T = (1 + \rho g)x_3 + f(x_1, x_2)$

where $f(x_1, x_2)$ is an arbitrary function of x_1, x_2.

Exercise 7.2 In the absence of body forces, do the stress components:

$$T_{11} = \alpha\left[x_2^2 + \upsilon\left(x_1^2 - x_2^2\right)\right]$$
$$T_{22} = \alpha\left[x_1^2 + \upsilon\left(x_2^2 - x_1^2\right)\right]$$
$$T_{33} = \alpha\upsilon\left[x_1^2 + x_2^2\right]$$
$$T_{12} = -2\alpha\upsilon x_1 x_2, T_{23} = T_{31} = 0$$

satisfy the equations of equilibrium ?

Unit : 6 ❑ Generalized Hooke's Law

6.1. STRAIN ENERGY

The work done in straining an elastic body from its unstrained state to the deformed or strained state by the surface force is transformed completely into the potential energy stored in that body. This potential energy which is due to deformation or strain only is called the strain energy or stress potential. We shall now discuss its existence.

From the first law of thermodynamics we have

$$\rho \frac{de}{dt} = T_{ij} d_{ij} - q_{i,i} + \rho h \qquad \text{......(1.1)}$$

where e = internal energy per unit mass

$$d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad \text{......(1.2)}$$

q_i = influx of heat by conduction per unit area per unit time.

h = rate per unit mass at which the heat energy by radiation is produced from internal sources.

We shall here consider the case of linear elastic solid for which only small change of shape can occur when it is subjected to forces of reasonable magnitude. In this case every stress component is a lienar function of all strain conponents. For such linear elastic solid one can neglect the heat conduction, and consequently the heat energy is produced from the internal sources only.

Therefore, we have from (1.1)

$$\rho \frac{de}{dt} = T_{ij} d_{ij} + \rho h \qquad \text{...... (1.3)}$$

Also, for small strains we have

$$d_{ij} = \dot{e}_{ij}$$

Consequently we have

$$T_{ij} \dot{e}_{ij} = \rho(\dot{e} - h) = \rho\left(\dot{e} - \dot{\bar{Q}}\right) \qquad \text{......(1.4)}$$

where $h = \frac{d\overline{Q}}{dt}$ ... (1.5)

$\overline{Q}$ being the quantity of heat per unit mass produced by internal sources at time t. Let us write

$$\rho_0 c = U \text{ and } \rho_0 \overline{Q} = Q \qquad \text{......(1.6)}$$

Here, U is the internal energy per unit volume of the unstrained body and Q is the quantity of heat produced from internal sources per unit volume of the unstrained state. Then, from (1.4),

$$T_{ij}\, \dot{e}_{ij} = \frac{\rho}{\rho_0}\left(\dot{U} - \dot{Q}\right) \qquad (1.7)$$

Now, $$\frac{\rho}{\rho_0} = \frac{dV_0}{dV} = \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} = \det\left|\frac{\partial x_i}{\partial X_j}\right|$$

$$= \det\left|u_{i,j} + \delta_{ij}\right| = 1 + u_{i,i} \quad \text{(for small displacements)}$$

Also, for small $u_{i,i}$, that is, for small displacement gradients we have $\frac{\rho}{\rho_0} = 1$, i.e., $\rho = \rho_0$. Therefore, the equation (1.7) becomes.

$$T_{ij}\, \dot{e}_{ij} = \dot{U} - \dot{Q} \qquad \text{......(1.8)}$$

Again, from the second law of thermodynamics we have

$$\dot{Q} = \frac{\partial Q}{\partial t} = T\frac{\partial S}{\partial t} \qquad \text{......(1.9)}$$

where T is the temperature per unit volume and S is the entropy. Now, we can identify two processes. The first one is the adiabatic process for which the change of state from one configuration to another takes place so rapidly that there is no time for the heat generated to be dissipated. For this we must have

$$\dot{Q} = 0 \qquad \text{......(1.10)}$$

and therefore $T_{ij}\, \dot{e}_{ij} = \dot{U}$ or, $T_{ij}\, de_{ij} = dU$ ......(1.11)

That is, the L.H.S. of (1.11) is an exact differential.

We can write $T_{ij}\, de_{ij} = dW$ ......(1.12) where W is the internal energy of the body. The other process is the isothermal process in which the change of state is sufficiently slow such that the heat generated has enough time for dissipation maintaining a constant temperature of the elastic solid. In this case the body remains in continued equilibrium of temperature with its surroundings. Therefore, for this case we have

$$\dot{T} = \frac{\partial T}{\partial t} = 0 \qquad \text{......(1.13)}$$

From (1.9), we have

$$\dot{Q} = T\frac{\partial S}{\partial t} + S\frac{\partial T}{\partial t} = \frac{\partial}{\partial t}(TS) \qquad \text{......(1.14)}$$

Using this equation we have from (1.8)

$$T\,\dot{e}_{ij} = \dot{U} - \frac{\partial}{\partial t}(TS) = \frac{\partial}{\partial t}(U - TS)$$

$$= \dot{F} \qquad \text{......(1.15)}$$

where $F = U - TS$ is the Helmholtz's free energy.

Thus, we have for isothermal process

$$T\,de_{ij} = dF \qquad \text{......(1.16)}$$

which shows the L.H.S. of this equation is an exact differential. Therefore, there exist a function W such that

$$T de_{ij} = dW \qquad \text{... ...(1.17)}$$

Here, W represents the Helmholtz's free energy per unit volume of the elastic medium.

Let us introduce the following notation :

$$T_{11} = T_1,\ T_{22} = T_2,\ T_{33} = T_3,\ T_{23} = T_{32} = T_4,$$

$$T_{31} = T_{13} = T_5,\ T_{12} = T_{21} + T_6$$

and $e_{11} = e_1$, $e_{22} = e_2$, $e_{33} = e_3$, $2e_{23} = 2e_{32} = e_4$, $2e_{31} = 2e_{13} = e_5$, $2e_{12} = 2e_{21} = e_6$.

With these notatoins for stresses and strains we can have from (1.12) and (1.17) a relation

$$T_{ij}\, de_{ij} = T_i de_i = dW \quad (i = 1, 2, ..., 6) \quad ......(1.18)$$

for both the adiabatic and isothermal processes. As $T_i de_i$ represents the work done per unit volume at a point by all surface forces, dW must be the work done per unit volume. From (1.18) we have

$$T_i\, de_i = \frac{\partial W}{\partial e_i} de_i$$

Therefore, $T_i = \dfrac{\partial W}{\partial e_i} \quad (i = 1, 2, ..., 6) \quad ......(1.19)$

Thus, for both the processes there exists a function. W given by (1.19). This fucntion W is called the stress potential or strain energy function per unit volume of the elastic body, as it is the potential energy per unit volume stored up in the elastic body by strain. The stress comonents are obtained as the partial derivatives of W with respect to the corresponding strain components.

6.2. GENERALIZED HOOKE'S LAW

For linear elastic solid every stress component is a linear function of all strain components. It is expressed by the following relation.

$$T_{ij} = b_{ij} + a_{ijkl}\, e_{kl}$$

The body is unstrained under no stresses, and therefore $e_{ij} = 0$ when $T_{ij} = 0$. Consequently, we have $b_{ij} = 0$. Therefore we have

$$T_{ij} = a_{ijkl}\, e_{kl} \; (i, j, k, l = 1, 2, 3) \quad ... \quad (2.1)$$

This relation between stress and strain is known as generalized Hooke's law for linear elastic solid. The coefficients a_{ijkl} are called elastic constants or elastic moduli of the elastic solid. They characterize the elastic properties of the body. The elastic solid is said to be inhomogeneous if these elastic moduli vary point to point of that body. But if these elastic moduli are unchanged throughout the medium then it will be called elasticaly homogeneous.

6.3. ISOTROPIC ELASTIC SOLID

A linearly elastic solid is known as to be isotropic if it has the symmetry of elastic properties in all directions. It means that the strain energy W is an invariant under all orthogonal transformations of co-ordinate axes. That is, W is independent of the orientation of coordinate axes and hence the symmetry in respect of all directions is maintained.

Let us write the elastic constants a_{ijkl} as the sum

$$a_{ijkl} = b_{ijkl} + c_{ijkl} \quad \text{......(3.1)}$$

where $b_{ijkl} = \frac{1}{2}\left(a_{ijkl} - a_{ijlk}\right) = -b_{ijlk}$ ......(3.2)

and $c_{ijkl} = \frac{1}{2}\left(a_{ijkl} + a_{ijlk}\right) = c_{ijlk}$ ......(3.3)

With these we have from generalized Hooke's law (2.1)

$$T_{ij} = b_{ijkl}\, e_{kl} + c_{ijkl}\, e_{kl} \quad \text{......(3.4)}$$

Now, $b_{ijkl}e_{kl} = b_{ijlk}e_{lk}$ (interchanging dummy suffixes)

$= -\, b_{ijkl}\, e_{kl}$ (using (3.2) and remembering that e_{kl} is symmetric)

$\therefore \quad b_{ijkl}\, e_{kl} = 0$

Consequently, from (3.4) we have

$$T_{ij} = c_{ijkl}\, e_{kl} \quad \text{......(3.5)}$$

T_{ij} and e_{ij} are second order tensor, and therefore c_{ijkl} form a tensor of order 4. For isotropic elastic medium the elastic constants c_{ijkl} remain the same under all orthogonal transformation of coordinate axes. Thus, for isotropic body c_{ijkl} must be an isotropic tenser of order four. That is, we can write.

$$c_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + \mu\delta_{ik}\, \delta_{jl} + \nu\delta_{il}\delta_{jk} \quad \text{......(3.6)}$$

where λ, μ, ν are constants. Using (3.3) we have

$$\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk} = \lambda\delta_{ij}\delta_{lk} + \mu\delta_{il}\delta_{jk} + \nu\delta_{ik}\delta_{jl}$$

or, $(\mu - \nu)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) = 0$

This relation is true for all values of the suffixes i, j, k, l.

We may put i = 1, k = 1, j = 2, l = 2 in the above relation to get $(\mu - \nu)(\delta_{11}\delta_{22} - \delta_{12}\delta_{21}) = 0$

or, $\mu - \nu = 0$, i.e., $\mu = \nu$

Therefore, from (3.6) we get

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \qquad ......(3.7)$$

Using this relation we have from (3.5)

$$T_{ij} = \left\{\lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right\}e_{kl}$$

$$= \lambda\delta_{ij}\,e_{kk} + \mu\left(e_{il}\delta_{jl} + e_{ik}\delta_{jk}\right)$$

$$= \lambda\delta_{ij}\,e_{kk} + \mu(e_{ij} + e_{ij})$$

or, $T_{ij} = \lambda\theta\,\delta_{ij} + 2\mu e_{ij}$ ......(3.8)

where $\theta = e_{kk}$ is the first strain invariant.

(3.8) are the constitutive equations or stress-strain relation for an isotropic linearly elastic body. We see that for such an elastic medium the number of elastic constants is only two, namely, λ and μ. The strain energy W is given by

$$W = \frac{1}{2}T_{ij}\,e_{ij} \qquad (\text{since } dW = T_{ij}\,de_{ij})$$

$$= \frac{1}{2}\left\{\lambda\theta\delta_{ij} + 2\mu e_{ij}\right\}e_{ij} = \frac{1}{2}\lambda\theta^2 + \mu e_{ij}e_{ij} \qquad ......(3.9)$$

Ex. 3.1. Show that the principal directions of strain at each point of a linearly elastic isotropic body are coincident with the principle directions of stress.

Proof : We take the principal directions of strain at a point of the medium as the co-ordinate axes. Then $e_{12} = e_{23} = e_{31} = 0$.

From the stress-strain relation for the linearly elastic isotropic medium we have

$$Tij = \lambda\,\theta\,\delta ij + 2\mu\,e_{ij}$$

$$T_{12} = 2\mu e_{12} = 0, \quad T_{23} = 2\mu e_{23} = 0, \quad T_{31} = 2\mu e_{31} = 0$$

Therefore, the co-ordinate axes are along the principal directions of stress. Thus, the principal directions of strain are coincident with the principal directions of stress for an isotropic elastic body.

6.4. ELASTIC MODULI FOR ISOTROPIC MEDIA

The two elastic constants λ and μ in the constitutive equation (3.8) are known as Lame's constants. From (3.8) it follows that

$$T_{ii} = \lambda\theta\delta_{ii} + 2\mu e_{ii} = 3\lambda\theta + 2\mu\theta$$

$$= (3\lambda + 2\mu)\theta$$

Since $T_{ii} = \Theta$, we have $\Theta = (3\lambda + 2\mu)\theta$

$$\text{or,} \quad \theta = \frac{\Theta}{3\lambda + 2\mu} \qquad \text{......(4.1)}$$

Therefore, from (3.8) we get

$$T_{ij} = \frac{\lambda\Theta}{3\lambda + 2\mu}\delta_{ij} + 2\mu e_{ij}$$

$$\text{or,} \quad e_{ij} = \frac{T_{ij}}{2\mu} - \frac{\lambda\Theta\delta_{ij}}{2\mu(3\lambda + 2\mu)} \qquad \text{......(4.2)}$$

Putting $i = 1, j = 1$, we have

$$e_{11} = \frac{T_{11}}{2\mu} - \frac{\lambda\Theta}{2\mu(3\lambda + 2\mu)} = \frac{T_{11}}{2\mu}\left\{1 - \frac{\lambda}{3\lambda + 2\mu}\right\} - \frac{\lambda(T_{22} + T_{33})}{2\mu(3\lambda + 2\mu)}$$

$$= T_{11}\frac{\lambda + \mu}{(3\lambda + 2\mu)\mu} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}(T_{22} + T_{33}) \qquad \text{......(4.3)}$$

Let us now suppose $T_{11} = T$, $T_{22} = T_{33} = T_{23} = T_{31} = T_{12} = 0$. This state of stress is possible in an elastic right circular cylinder the axis of which is parallel to x_1 axis and subjected to an uniform longitudinal axial tensile loading to both of its ends. Also, the above state of stress statisfies the equilibrium equations in absence of body forces at every point in the interior of the cylindrical elastic medium and also satisfies the stress-free boundary condition on its lateral surface.

Then, from (4.2) and (4.3),

$$\left.\begin{aligned} e_{11} &= T\frac{\lambda+\mu}{(3\lambda+2\mu)\mu}, e_{22} = e_{33} = -\frac{\lambda T}{2u(3\lambda+2\mu)} \\ e_{23} &= e_{31} = e_{21} = 0 \end{aligned}\right\} \quad \ldots\ldots(4.4)$$

We see that the ratio of tensile stress T_{11} to the longitudinal extension e_{11}, that is, $\frac{T_{11}}{e_{11}} = \frac{T}{e_{11}} = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ = constant. This constant is called Young's modulus or modulus of elasticity, and is denoted by by E. Thus,

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \quad \ldots\ldots(4.5)$$

Again, the ratio of lateral contraction to longitudinal extension, that is, $\frac{-e_{22}}{e_{11}} = \frac{\lambda}{2(\lambda+\mu)}$ = constant.

This constant is called Poisson's ratio, and is denoted by σ. Thus,

$$\sigma = \frac{\lambda}{2(\lambda+\mu)} \quad \ldots\ldots(4.6)$$

From (4.5) and (4.6) it is easy to see that

$$\left.\begin{aligned} \lambda &= \frac{E\sigma}{(1+\sigma)(1-2\sigma)} \\ \mu &= \frac{E}{2(1+\sigma)} \end{aligned}\right\} \quad \ldots\ldots(4.7)$$

Next we consider the state of stress given by

$$T_{23} = S = \text{constant}, \quad T_{11} = T_{22} = T_{33} = T_{12} = T_{31} = 0 \quad \ldots\ldots(4.8)$$

This state of stress is possible in a deformed long rectangular parallelopiped of square cross-section OABC (Fig. 7), which is sheared in the plane containing OA and OC by a shearing stress of magnitude S acting per unit area on the side CB. The stress S will tend to slide the planes of the material originally perpendicular to OC, the x_3-axis, in a direction parallel to OA, the x_2-axis such that the right angle OBC is diminished by an angle ϕ.

The above state of stress satisfies the equations of equilibrium in absence of body force at each point in the interior and with the boundary condition at the surface. We have, from (4.2) and (4.8),

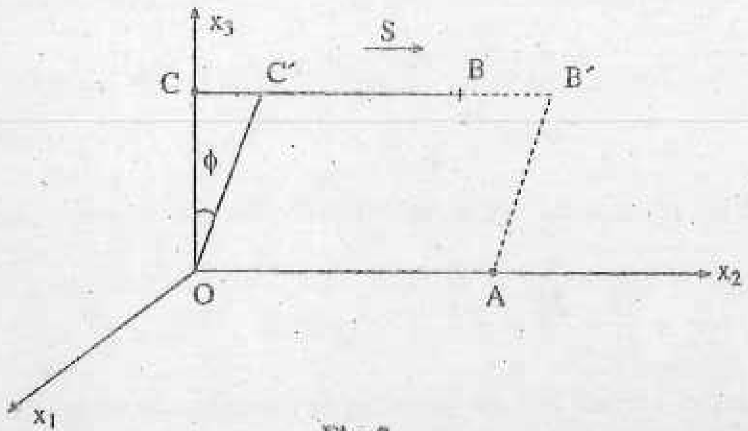


Fig. 7

$$e_{23} = \frac{T_{23}}{2\mu} = \frac{S}{2\mu}, \quad e_{11} = e_{22} = e_{33} = e_{31} = e_{12} = 0. \quad \text{......(4.9)}$$

Also, we know that $2e_{23} = \phi$

$$\therefore \quad \frac{S}{\phi} = \frac{S}{2e_{23}} = \mu$$

This ratio is known as shear modulus or modulus of rigidity, and it is identical with the Lame's constant μ.

Let us consider now an elastic body subjected to a hydrostatic stress p distributed over its surface. Due to this hydrostatic stress the volume of the body is diminished. The state of stress possible in such a deformed body is given by

$$\left.\begin{array}{l} T_{11} = T_{22} = T_{33} = -p = \text{constant}, \\ T_{23} = T_{31} = T_{12} = 0 \end{array}\right\} \quad \text{......(4.10)}$$

The state of stress statisfies the equations of equilibrium in the interior of body with the boundary condition on its surface. Actually, if $T_i^{(n)}$ (i = 1, 2, 3) represent the stress vector acting on the surface with normal $\bar{n}$, then $T_i^{(n)} = -pn_i$ (i = 1, 2, 3), or $T_{ij}\, n_j = -pn_i$ at each point on the surface. This condition on the boundary is satisfied by the state of stress given in (4.10). From (4.2) and (4.10) we have

$$\left.\begin{aligned} e_{11} = e_{22} = e_{33} &= \frac{-p}{2\mu} + \frac{3\lambda p}{2\mu(3\lambda + 2\mu)} = -\frac{p}{3\lambda + 2\mu} \\ e_{12} = e_{23} = e_{31} &= 0 \end{aligned}\right\} \quad \text{......(4.11)}$$

Now, θ = cubical dilatation. Therefore, the decrease in volume per unit volume

$$= -\theta = -e_{ii} = -\left(e_{11} + e_{22} + e_{33}\right)$$

$$= \frac{3p}{3\lambda + 2\mu}$$

We see that the ratio of the hydrostatic stress to the decrease in volume per unit volume, that is,

$$\frac{p}{-\theta} = \frac{3\lambda + 2\mu}{3} = \lambda + \frac{2\mu}{3} = \text{constant}$$

This constant is known as bulk modulus or modulus of compression, and is denoted by K. Thus,

$$K = \lambda + \frac{2\mu}{3} \quad \text{......(4.12)}$$

It is easy to see that

$$K = \frac{E}{3(1 - 2\sigma)} \quad \text{......(4.13)}$$

For positive K and E we must have $0 < \sigma < \frac{1}{2}$. Also, we see that $\lambda > 0$, $\mu > 0$ (from (4.7))

From (4.5) and (4.6) we can have

$$\frac{1+\sigma}{E} = \frac{1}{2\mu} \text{ and } \frac{\sigma}{E} = \frac{\lambda}{2\mu(3\lambda + 2\mu)}$$

Then the strain-stress relation (4.2) becomes.

$$e_{ij} = \frac{1+\sigma}{E} T_{ij} - \frac{\sigma}{E} \Theta \delta_{ij} \quad \text{......(4.14)}$$

From this relation we get

$$\theta = e_{ii} = \frac{1+\sigma}{E} T_{ii} - \frac{3\sigma}{E} \Theta = \frac{1+\sigma}{E} \Theta - \frac{3\sigma}{E} \Theta$$

$$= \frac{1-2\sigma}{E}\Theta = \frac{\Theta}{3K} \quad \text{(using (4.13))}$$

or, $$\frac{\Theta}{3} = K\theta \qquad \text{......(4.15)}$$

6.5. BELTRAMI-MICHELL COMPATIBILITY EQUATIONS

The equations of compatibility for strains are given in the previous chapter (unit). They are given by

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0 \qquad \text{......(5.1)}$$

From these equations it follows that (setting l = k and summing over k)

$$e_{ij,kk} + e_{kk,ij} - e_{ik,jk} - e_{jk,ik} = 0 \qquad \text{......(5.2)}$$

From (4.14) we have

$$e_{ij} = \frac{1+\sigma}{E}\left(T_{ij} - \frac{\sigma}{1+\sigma}\Theta\delta_{ij}\right) \qquad \text{......(5.3)}$$

Substituting (5.3) into (5.2) we get

$$T_{ij,kk} + T_{kk,ij} - T_{ik,jk} - T_{jk,ik} = \frac{\sigma}{1+\sigma}\left(\delta_{ij}\Theta_{,kk} + \delta_{kk}\Theta_{,ij} - \delta_{ik}\Theta_{,jk} - \delta_{jk}\Theta_{,ik}\right)$$

or, $$\nabla^2 T_{ij} + \Theta_{,ij} - \frac{\sigma}{1+\sigma}\delta_{ij}\nabla^2\Theta - \frac{3\sigma}{1+\sigma}\Theta_{,ij} + \frac{\sigma}{1+\sigma}\Theta_{,ji} + \frac{\sigma}{1+\sigma}\Theta_{,ij} = T_{ik,jk} + T_{jk,ik}$$

or, $$\nabla^2 T_{ij} + \frac{1}{1+\sigma}\Theta_{,ij} - \frac{\sigma}{1+\sigma}\delta_{ij}\nabla^2\Theta = T_{ik,jk} + T_{jk,ik} \qquad \text{......(5.4)}$$

From the equations of equilibrium we have

$$T_{ik,k} + \rho F_i = 0$$

Therefore, $T_{ik,kj} = -\rho F_{i,j}$

Similarly, we get $T_{jk,ik} = -\rho F_{j,i}$

Therefore, from (5.4),

$$\nabla^2 T_{ij} + \frac{1}{1+\sigma}\Theta_{,ij} - \frac{\sigma}{1+\sigma}\delta_{ij}\nabla^2\Theta = -\rho\left(F_{i,j} + F_{j,i}\right) \qquad \text{......(5.5)}$$

Setting j = i and summing over i we have from (5.5)

$$\nabla^2\Theta + \frac{1}{1+\sigma}\nabla^2\Theta - \frac{3\sigma}{1+\sigma}\nabla^2\Theta = -2\rho \operatorname{div}\vec{F}$$

or, $$\frac{1-\sigma}{1+\sigma}\nabla^2\Theta = -2\rho \operatorname{div}\vec{F}$$

or, $$\nabla^2\Theta = -2\frac{1+\sigma}{1-\sigma}\rho \operatorname{div}\vec{F} \qquad ......(5.6)$$

Substituting (5.6) into (5.5) we have

$$\nabla^2 T_{ij} + \frac{1}{1+\sigma}\Theta_{,ij} = -\frac{2\sigma}{1-\sigma}\delta_{ij}\rho \operatorname{div}\vec{F} - \rho\left(F_{i,j} + F_{j,i}\right) \qquad ......(5.7)$$

These six independent equations are known as Beltrami-Michell compatibility equations for stresses.

Unit : 7 ❑ Fluid Media

7.1. FLUID AND ITS BEHAVIOUR

If a very small applied external force to a material medium can cause a continuous shear deformation of relative sliding such that its constituent particles become freely mobile, then the medium is called "fluid", and the continuous shear deformation of it is known as the flow of fluid. Unlike the solid elastic medium, fluid has no stress-free state to which it eventually returns if the external force is withdrawn. Consequently, every configuration can be regarded as the reference configuration. The continuous flow of fluid changes the shape of the fluid. It has no definite shape, but takes the shape of the container into which it is placed.

The fluid in motion exerts on any adjacent layer moving with different velocity some kind of frictional resistance to alternations of form in the form of shearing stress in the tangential plane in addition to the normal stress in order to accelerate or dissipate its state of relative motion. For fluid at rest these shearing stresses have no role to play. The relative motion of fluid layers gives rise to these shearing stresses, and these stresses vanish when the rate of deformation or the rate of change of strain is zero. This property of the "fluid in motion" to exert shearing stress on the adjacent layers in resisting their sliding motion under the action of very small shearing force is the viscosity of the fluid, and the fluid having this property is called viscous fluid. If the stress exerted by a viscous fluid is a linear function of the rate of change of strain such that this stress is zero when the rate of strain is zero, then the fluid is known as Newtonian Viscous fluid or linearly Viscous fluid. On the other hand, if the fluid is incapable of exerting any shearing stresses on the adjacent layers in its contact to resist its shearing movement under a very small shearing force then it is called a perfect fluid.

7.2. LAGRANGIAN AN AND EULERIAN METHODS OF DESCRIPTIONS :

We have discussed Lagrangian and Eulerian methods of description in Unit 3. In Eulerian method of description for fluid motion we consider the velocity of the material point. $\vec{v} = (v_1, v_2, v_3)$ is the velocity of the material point occupying the spatial point (x_1, x_2, x_3) at time t. It is given by

$$v_i = f_i(x_1, x_2, x_3, t) \quad (= 1, 2, 3) \qquad \text{......(2.1)}$$

On the other hand, in the Lagrangian method the motion of each individual material point of fixed identity is described for all time by following its motion throughout the course. The initial position (X_1, X_2, X_3) may be taken as the fixed identification of the material point. One can now transit from the Eulerian method to the Lagrangian method by writing $\frac{dx_i}{dt}$ for v_i in the equation (2.1) to obtain

$$\frac{dx_i}{dt} = f_i(x_1, x_2, x_3, t) \quad (i = 1, 2, 3) \qquad \text{......(2.2)}$$

If we integrate the above three equations then three constants of integration will appear. One can take these three constants to be the three co-ordinates X_1, X_2, X_3 which identify the material point in the Lagrangian method of description. Thus, as the solutions of (2.2) we can have

$$x_i = F_i(X_1, X_2, X_3, t) \quad (i = 1, 2, 3) \qquad \text{......(2.3)}$$

Obviously, these equations specify the motion in the Lagrangian method.

7.3. STREAM LINE :

A stream line is a curve drawn in the fluid medium at any given instant of time such that the tangent at each point of it is coincident with the instantaneous direction of the velocity of the material point at that point. Let $\vec{v} = (v_1, v_2, v_3)$ be the velocity of the particle (material point) at any point $P(x_1, x_2, x_3)$ on the stream line at time t (Fig 3.1). Let $\vec{r}$ be the position vector of P.

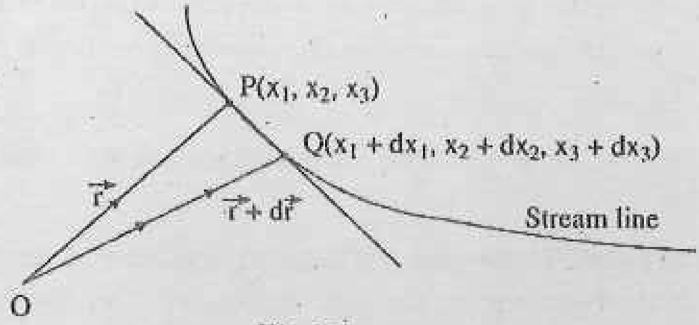


Fig. 3.1

If $Q(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$ is a neighbouring point of P on the stream line then the straight line PQ will be the tangent to the stream line when Q is very closed to P. That is, dx_i $(i = 1, 2, 3)$ are infinitesimal quantities. If $\vec{r} + d\vec{r}$ is the position vector of Q, then $\vec{PQ} = d\vec{r} = (dx_1, dx_2, dx_3)$. Since $\vec{v}$ is in the direction of $d\vec{r}$ at the instant t, we must have $\vec{v} = kd\vec{r}$ where k is a constant.

Consequently, we have, at the instant t,

$$\frac{dx_1}{v_1(x_1, x_2, x_3, t)} = \frac{dx_2}{v_2(x_1, x_2, x_3, t)} = \frac{dx_3}{v_3(x_1, x_2, x_3, t)} \qquad (3.1)$$

This equation is the differential equation of the stream line at a given instant t. The velocity components $v_i = (i = 1, 2, 3)$ are evaluated at the instant of time t, and thus t, here, plays the role of a parameter whereas the velocity remains the function of the position coordintes. Obviously, the stream lines are changed from instant to instant.

The stream line is different from the path line of a particle. If fact, a path line of a particle is the curve in the fluid followed by it during its entire motion. If $\vec{v}$ is the velocity of the particle at any point (x_1, x_2, x_3) whose position vector is $\vec{r}$, then we have $\vec{v} = \frac{d\vec{r}}{dt}$. As the point $\vec{r} = (x_1, x_2, x_3)$ lies on the path line at time t we must have

$$v_i = \frac{dx_i}{dt} \quad (i = 1, 2, 3) \qquad (3.2)$$

$$\text{or,} \quad \frac{dx_1}{v_1(x_1, x_2, x_3, t)} = \frac{dx_2}{v_2(x_1, x_2, x_3, t)} = \frac{dx_3}{v_3(x_1, x_2, x_3, t)} = dt \qquad (3.3)$$

as the differential equation of the path line. Here, $V_i(x_1, x_2, x_3, t)$ $(i = 1, 2, 3)$ are functions of space coordinates and time. If (X_1, X_2, X_3) are the initial coordinates of the particle, that is, at $t = 0$, identifying the material point then the integral curve of (3.3) must passes through it. For steady motion, that is, when $\frac{\partial \vec{v}}{\partial t} = 0$ or, $\vec{v} = \vec{v}(x_1, x_2, x_3)$ the path line coincides with the stream line, as it is evident from (3.1) and (3.3).

7.4. ROTATIONAL AND IRROTATIONAL MOTION : VELOCITY POTENTIAL

Let us consider two neighbouring particles in "fluid in motion" at the positions P and Q having position vectors $\vec{r}$ and $\vec{r} + d\vec{r}$ respectively, where $\vec{r} = (x_1, x_2, x_3)$ and $\vec{r} + d\vec{r} = (x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$. Let the velocities of the particles at P and Q be respectively $\vec{v}$ and $\vec{v} + d\vec{v}$.

Then in the Eulerian method we have

$$\vec{v} = \vec{v}(x_1, x_2, x_3, t) \quad \text{or,} \quad v_i = v_i(x_1, x_2, x_3, t)$$

$$\vec{v} + d\vec{v} = \vec{v}(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3, t)$$

$$\text{or,} \quad v_i + dv_i = v_i(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3, t)$$

$$i = 1, 2, 3 \qquad \ldots\ldots(4.1)$$

Now, we have $v_i + dv_i = v_i(x_1, x_2, x_3, t) + \dfrac{\partial v_i}{\partial x_j} dx_j + \ldots\ldots$

(by Taylor's expansion)

Therefore,

$$dv_i = \frac{\partial v_i}{\partial x_j} dx_j = v_{i,j}\, dx_j \qquad \ldots\ldots(4.2)$$

(neglecting the terms containing squares and higher powers of the small differentials dx_j)

or, we can write

$$dv_i = d_{ij}dx_j + w_{ij}dx_j = dv_i^{(1)} + dv_i^{(2)} \qquad \ldots\ldots(4.3)$$

with $dv_i^{(1)} = d_{ij}\, dx_j$ ......(4.4)

and $dv_i^{(2)} = w_{ij}dx_j$ ......(4.5)

where

$$d_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right) = d_{ji} \qquad \ldots\ldots(4.6)$$

= symmetric strain-rate tensor of order 2 and

$$w_{ij} = \frac{1}{2}\left(v_{i,j} - v_{j,i}\right) = -w_{ji} \qquad \text{......(4.7)}$$

= skewsymmetric tensor of order 2.

One can form a vector w_i by setting

$$w_i = \epsilon_{ijk}\, w_{kj} \qquad \text{......(4.8)}$$

where ϵ_{ijk} is defined as

$\epsilon_{ijk} = 0$ if any two of i, j, k are equal

$= +1$ if i, j, k are even permutation of 1, 2, 3

$= 1$ if i, j, k are odd permutation of 1, 2, 3

Then

$$\epsilon_{ijk}\, w_i = \epsilon_{ijk}\, \epsilon_{ipq}\, w_{qp}$$

$$= \left(\delta_{jp}\, \delta_{kp} - \delta_{jq}\delta_{kp}\right) w_{qp}$$

$$= w_{kj} - w_{jk} = 2w_{kj} \qquad (\sin\ w_{jk} = -w_{kj})$$

Therefore, $w_{kj} = \frac{1}{2}\, \epsilon_{ijk}\, w_i$ ......(4.9)

Then, form (4.5) we have

$$dv_k^{(2)} = w_{kj}\, dx_j = \frac{1}{2}\, \epsilon_{ijk}\, w_i\, dx_j \ \text{(using (4.9))} \qquad \text{......(4.10)}$$

or, in vector notation

$$d\,\vec{v}^{(2)} = \frac{1}{2}\vec{w} \times d\,\vec{r} \qquad \text{......(4.11)}$$

where $d\,\vec{r} = \overrightarrow{PQ}$

Again, from (4.7) and (4.8) we get

$$w_i = \epsilon_{ijk}\, w_{kj} = \frac{1}{2}\, \epsilon_{ijk}\left(v_{k,j} - v_{j,k}\right)$$

$$= \frac{1}{2}\left(\epsilon_{ijk}\, v_{k,j} - \epsilon_{ikj}\, v_{k,j}\right) \qquad \text{(intechanging dummy suffixes)}$$

$$= \frac{1}{2}\left(\epsilon_{ijk}\, v_{k,j} + \epsilon_{ijk}\, v_{k,j}\right) = \epsilon_{ijk}\, v_{k,j}$$

That is,

$$\vec{w} = \text{rot}\,\vec{v} \equiv \text{curl}\,\vec{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix} \qquad \text{......(4.12)}$$

Consequently,

$$d\,\vec{v}^{(2)} = \frac{1}{2}\,\text{curl}\,\vec{v} \times d\,\vec{r} \qquad \text{......(4.13)}$$

The vector $\vec{w}$ = curl $\vec{v}$ is known as vorticity vector and the tensor w_{ij} is called spin tensor. Thus, we see that the velocity of Q, that is, the velocity of the neighbourhood of P is given by

$$v_i + dv_i = v_i + dv_i^{(1)} + dv_i^{(2)} \qquad \text{......(4.14)}$$

Obviously, if it comprising of three parts, namely, (i) translational motion with the velocity $\vec{v}$, the same as that of P.

(ii) motion that is causing a rate of change in relative position, given by $d_{ij}\,dx_j$,

and (iii) a rigid body rotational motion with angular velocity $\frac{1}{2}$ curl $\vec{v}$.

Now, if the state of motion in the neighbourhood of any particle in the fluid continuum is such that the rotational part of the motion vanishes, that curl $\vec{v} = 0$ at every point of the continuum, then the motion is said to be "irrotational". On the contrary, if the state of motion has the rotational part, that is, the vorticity vector $\vec{w}$ = curl $\vec{v}$ is non-zero, then the motion is "rotational" or "vortex motion".

Now, there may exist a scalar function ϕ such that the velocity components can be expressible as

$$v_i = -\frac{\partial \phi}{\partial x_i} \quad (i = 1, 2, 3) \qquad \text{......(4.15)}$$

or, the velocity as

$$\vec{v} = -\,\text{grad}\;\phi \equiv -\nabla\phi \qquad \text{......(4.16)}$$

For this case, we call the function ϕ the "vector potential."

If this function ϕ exists such that (4.15) or (4.16) holds good then we have

$$\text{curl } \vec{v} = -\text{ curl grad } \phi = -\begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ \dfrac{\partial \phi}{\partial x_1} & \dfrac{\partial \phi}{\partial x_2} & \dfrac{\partial \phi}{\partial x_3} \end{vmatrix} = 0$$

that is, the motion is irrotational. Conversely, we can show that if the motion is irrotational then the velocity potential ϕ must exist. To show this, we use Stokes' theorem which states that if $\vec{F}$ is a continuously differentiable vector point function and S be an open two-sided surface bounded by a simple closed curve Γ then

$$\int_S \vec{n}.\text{curl } \vec{F}\, ds = \oint_\Gamma \vec{F}.d\vec{r} \qquad \text{......(4.17)}$$

where $\vec{n}$ is the unit outward-drawn normal vector on the surface element dS of S and $d\vec{r}$ is the directed element of the curve. This theorem holds good for the simply-connected negion for which the curve Γ and the surface S entirely lie within it.

Now, let us consider a closed curve Γ and a surface Σ having the curve Γ as its rim, the boundary curve, entirely within the fluid continuum (Fig. 4.1).

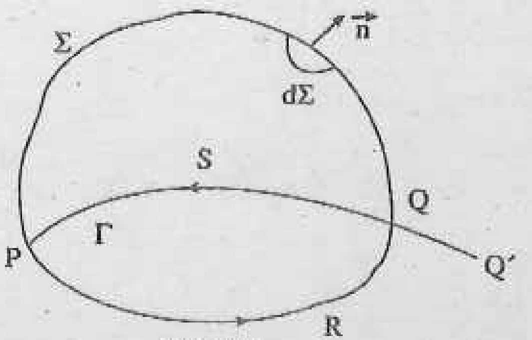


Fig. 4.1

Let $\vec{v}$ be the velocity, and $d\vec{r}$ be the directed element of the curve Γ which is PRQSP. Then by Stokes' theorem (4.17) we have

$$\oint_\Gamma \vec{v}.d\vec{r} = \int_\Sigma \vec{n}.\text{ curl } \vec{v} \qquad \text{......(4.18)}$$

Now, if the fluid motion is irrotational then curl $\vec{v} = 0$, and consequently we have

$$\oint_{\Gamma} \vec{v}.d\vec{r} \equiv \oint_{PRQSP} \vec{v}.d\vec{r} = 0$$

or, $$\oint_{PRQ} \vec{v}.d\vec{r} + \oint_{QSP} \vec{v}.d\vec{r} = 0$$

or, $$\oint_{PRQ} \vec{v}.d\vec{r} - \oint_{PSQ} \vec{v}.d\vec{r} = 0 \quad \text{or,} \quad \oint_{PRQ} \vec{v}.d\vec{r} = \oint_{PSQ} \vec{v}.d\vec{r}$$

This shows that the values of the integrals along the curves PRQ and PSQ from the point P to the point Q are the same. In other words, the value of the integral does not depend on the path of integration from P to Q, and consequently it depends only on the end points P and Q. Now, if P is a fixed point then the integral will depend on the position of Q only. Therefore, there must be a function ϕ such that

$$\oint_{PRQ} \vec{v}.d\vec{r} = \int_{P}^{Q} \vec{v}.d\vec{r} = -\phi(Q) \qquad ......(4.19)$$

Now, let Q′ be a neighbouring point of Q, and the point Q′ lies on a curve joining P and Q. Then we have

$$\int_{P}^{Q'} \vec{v}.d\vec{r} = -\phi(Q') \qquad ......(4.20)$$

Therefore, from (4.19) and (4.20),

$$\int_{P}^{Q} \vec{v}.d\vec{r} - \int_{P}^{Q'} \vec{v}.d\vec{r} = \phi(Q') - \phi(Q)$$

or, $$-\int_{Q}^{Q'} \vec{v}.d\vec{r} = \int_{Q}^{Q'} d\phi$$

Consequently, we have

$\vec{v}.d\vec{r} = -d\phi$ at every point in the continuum,

or, we have $\vec{v}.d\vec{r} = -\nabla\phi.d\vec{r} = -\text{grad}\,\phi.d\vec{r}$

Since the element $d\vec{r}$ is arbitrary we must have $\vec{v} = -\text{grad}\,\phi$

ensuring the existence of the velocity potential. Thus, the necessary and sufficient condition for the existence of the velocity potential is that the motion must be irrotational. Therefore, the motion characterized by the velocity potential and the irrotational motion are the same.

For the rotational or vortex motion, a curve drawn in the fluid medium at any given instant of time such that the tangent at every point of it gives the instantaneous direction of the vorticity vector $\vec{w}$ at that point is called the vortex line. If dx_1, dx_2, dx_3 are the components of the directed element $d\vec{r}$ of the curve at any point then $\vec{w} = k\,d\vec{r}$, and consequently, we have the following differential equation of the vortex line at an instant of time t :

$$\frac{dx_1}{w_1} = \frac{dx_2}{w_2} = \frac{dx_3}{w_3} \qquad \text{......(4.21)}$$

Since $\vec{w} = \text{curl}\,\vec{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix}$

we have, from (4.21),

$$\frac{dx_1}{\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}} = \frac{dx_2}{\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}} = \frac{dx_3}{\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}} \qquad \text{......(4.22)}$$

Ex. 4.1. Find the stream line and path line of a continuum particle for the velocity field given by

$$v_1 = \frac{x_1^2}{1+t^2}, \quad v_2 = x_2^2, \quad v_3 = 0$$

Solution : The differential equation of the stream line at a given instant t of time is

$$\frac{dx_1}{\frac{x_1^2}{1+t^2}} = \frac{dx_2}{x_2^2} = \frac{dx_3}{0} \quad \text{with t being a fixed parameter}$$

Integrating we have (keeping in mind that t is a constant)

$$-\left(1+t^2\right)\frac{1}{x_1}+C_1=-\frac{1}{x_2}$$ and $x_3=C_2$ where C_1 and C_2 are constants of integration. Then the stream line is given by

$$\left(1+t^2\right)\frac{1}{x_1}-\frac{1}{x_2}=C_1,\ x_3=C_2$$

The differential equation for path line is

$$\frac{dx_1}{\frac{x_1^2}{1+t^2}}=\frac{dx_2}{x_2^2}=\frac{dx_3}{0}=dt \quad \text{where t is now a variable}$$

or, $$\frac{dx_1}{x_1^2}=\frac{dt}{1+t^2},\quad \frac{dx_2}{x_2^2}=dt \quad \text{and } dx_3=0$$

or, integrating, $-\frac{1}{x_1}+C_3=\tan^{-1}t,\ C_4-\frac{1}{x^2}=t$ and $x_3=C_5$

where C_3, C_4 and C_5 are constants of integration. Eliminating t we have the following equation for the path line :

$$\tan\left(C_3-\frac{1}{x_1}\right)=C_4-\frac{1}{x_2},\ x_3=C_5$$

Ex. 4.2. For the velocity field given by $v_1 = kx_3$, $v_2 = kx_3$, $v_3 = k(x_1 + x_2)$ show that the motion is irrotational. Find the velocity potential and the stream line.

Solution :

$$\operatorname{curl}\vec{v}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ kx_3 & kx_3 & k(x_1+x_2) \end{vmatrix} = i\left\{\frac{\partial}{\partial x_2}\left[k(x_1+x_2)\right]-\frac{\partial}{\partial x_3}(kx_3)\right\} + j\left\{\frac{\partial(kx_3)}{\partial x_3}-\frac{\partial}{\partial x_1}\left[k(x_1+x_2)\right]\right\} + k\left\{\frac{\partial}{\partial x_1}(kx_3)-\frac{\partial}{\partial x_2}(kx_3)\right\}$$

$$=i\{k-k\}+j\{k-k\}+k\,.\,0$$

$$=0$$

$\therefore$ the motion is irrotational.

If ϕ is the velocity potential

$$-\frac{\partial\phi}{\partial x_1} = v_1 = kx_3, \quad -\frac{\partial\phi}{\partial x_2} = v_2 = kx_3, -\frac{\partial\phi}{\partial x_3} = k(x_1 + x_2)$$

$$\therefore \quad d\phi = \frac{\partial\phi}{\partial x_1}dx_1 + \frac{\partial\phi}{\partial x_2}dx_2 + \frac{\partial\phi}{\partial x_3}dx_3$$

$$= -kx_3\,dx_1 - kx_3dx_2 - k(x_1 + x_2)dx_3$$

$$= -k\{x_3dx_1 + x_1dx_3 + x_3dx_2 + x_2dx_3\}$$

$$= -k\{d(x_3x_1) + d(x_3x_2)\} = -kd(x_3x_1 + x_3x_2)$$

$$\therefore \quad \phi = -k(x_3x_1 + x_3x_2) + A \quad (A = \text{constant})$$

$$= -kx_3(x_1 + x_2) + A$$

Stream line is given by

$$\frac{dx_1}{kx_3} = \frac{dx_2}{kx_3} = \frac{dx_3}{k(x_1 + x_2)}$$

$$\therefore \quad x_3dx_3 = (x_1 + x_2)dx_1 \text{ and } dx_1 = dx_2$$

Integrating we have $x_1 = x_2 + C_1$, C_1 is a constant

Also, $x_3dx_3 = (x_1 + x_1 - C_1)dx_1 = (2x_1 - C_1)dx_1$

$$\therefore \quad \frac{1}{2}x_2^3 = x_1^2 - C_1x_1 + C_2, \; C_2 \text{ is a constant}$$

$\therefore$ eqn. of stream line : $x_1 - x_2 = C_1$, $x_2^3 = 2x_1^2 - 2C_1x_1 + 2C_2$

Ex. 4.3. For the velocity field $v_1 = \frac{ax_1}{1+t}$, $v_2 = \frac{2ax_2}{1+t}$, $v_3 = \frac{3ax_3}{1+t}$, determine the stream lines and path lines.

Solution : stream line is given by the differential equation

$$\frac{dx_1}{\frac{ax_1}{1+t}} = \frac{dx_2}{\frac{2ax_2}{1+t}} = \frac{dx_3}{\frac{3ax_3}{1+t}} \text{ where t is a fixed parameter}$$

Integrating with have $\log x_1 = \frac{1}{2}\log x_2 + \text{constant}$, and

$$\log x_1 = \frac{1}{3}\log x_3 + \text{constant}$$

$\therefore\; x_1 = C_1\, x_2^{1/2} = C_2 x_3^{1/3}$ is the stream line where C_1 and C_2 are constants.

Path line is given by

$$\frac{dx_1}{\frac{ax_1}{1+t}} = \frac{dx_2}{\frac{2ax_2}{1+t}} = \frac{dx_3}{\frac{3ax_3}{1+t}} = dt \text{ where t is now a variable.}$$

$$\therefore \frac{dx_1}{x_1} = \frac{adt}{1+t}, \frac{dx_2}{x_2} = \frac{2a\,dt}{1+t}, \frac{dx_3}{x_3} = \frac{3a\,dt}{1+t}$$

Integrating we have

$$\log x_1 = a\log(1+t) + \text{const.}, \log x_2 = 2a\log(1+t) + \text{constant}$$

$$\log x_3 = 3a\log(1+t) + \text{constant}$$

$$\therefore x_1 = C_3(1+t)^a, \quad x_2 = C_4(1+t)^{2a}, \quad x_3 = C_5(1+t)^{3a}$$

where C_3, C_4 and C_5 are constants of integratoin. We can eliminate t as follows :

$$\frac{x_1^2}{C_3^2} = (1+t)^{2a} = \frac{x_2}{C_4}, \frac{x_1^3}{C_3^3} = (1+t)^{3a} = \frac{x_3}{C_5}$$

or, $x_1^2 = \frac{C_3^2}{C_2} x_2$, $x_1^3 = \frac{C_3^3}{C_5} x_3$ gives the equation of the path line. Note that stream line and the path line are identical in this case.

Ex. 4.4. For the velocity field $v_1 = \frac{3x_1^2 - r^2}{r^5}$, $v_2 = \frac{3x_1x_2}{r^5}$, $v_3 = \frac{3x_1x_3}{r_5}$ where $r^2 = x_1^2 + x_2^2 + x_3^2$, show that the motion is irrotational. Find the velocity potential.

Solution : $\text{curl } \vec{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix}$

$$\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} = 3x_1x_3\left(-\frac{5}{r^6}\right)\frac{\partial r}{\partial x_2} - 3x_1x_2\left(-\frac{5}{r^6}\right)\frac{\partial r}{\partial x_3}$$

Now, $\frac{\partial r}{\partial x_2} = \frac{\partial}{\partial x_2}\left\{\left(x_1^2 + x_2^2 + x_2^3\right)^{1/2}\right\} = \frac{1}{2}\frac{2x_2}{\left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}} = \frac{r_2}{r}$

Similarly, $\frac{\partial r}{\partial x_3} = \frac{x_3}{r}$

$$\therefore \quad \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} = -\frac{15}{r^7} x_1 x_2 x_3 + \frac{15}{r^7} x_1 x_2 x_3 = 0$$

Similarly, $\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} = 0$ and $\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = 0$

Therefore, curl $\vec{v} = 0$, and the motion is irrotaional.

Let ϕ be the velocity potential. Then

$$d\phi = \frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{\partial x_2} dx_2 + \frac{\partial \phi}{\partial x_3} dx_3 = -v_1 dx_1 - v_2 dx_2 - v_3 dx_3$$

$$= -\left\{ \frac{3x_1^2 - r^2}{r^5} dx_1 + \frac{3x_1 x_2}{r^5} dx_2 + \frac{3x_1 x_3}{r^5} dx_3 \right\}$$

$$= -\frac{1}{r^5} \left\{ 3x_1^2 dx_1 + 3x_1 x_2 dx_2 + 3x_1 x_3 dx_3 - r^2 dx_1 \right\}$$

$$= -\frac{1}{r^5} \left\{ 3x_1 \left(x_1 dx_1 + x_2 dx_2 + x_3 dx_3 \right) - r^2 dx_1 \right\}$$

$$= -\frac{1}{r^5} \left\{ \frac{3x_1}{2} d\left(x_1^2 + x_2^2 + x_3^2 \right) - r^2 dx_1 \right\} = -\frac{1}{r^5} \left\{ \frac{3x_1}{2} d\left(r^2 \right) - r^2 dx_1 \right\}$$

$$= -\left\{ \frac{3x_1}{2r^5} d\left(r^2 \right) - \frac{1}{r^3} dx_1 \right\} = \frac{dx_1}{r^3} - \frac{3x_1}{r^4} dr = d\left(\frac{x_1}{x^3} \right)$$

$$\therefore \quad \phi = \frac{x_1}{r^3} = \frac{x_1}{\left(x_1^2 + x_2^2 + x_3 \right)^{3/2}}$$

Ex.4.5. Show that for the velocity field given by $v_1 = ax_3 - bx_2$, $v_2 = bx_1 - cx_3$, $v_3 = cx_2 - ax_1$ the motion is rotational.

Find the vortex lines.

Solution : Curl $\vec{v} = i\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \right) + j\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \right) + k\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right)$

or, curl $\vec{v} = i(c + c) + j(a + a) + k(b + b)$

$$= 2(ci + aj + bk) \neq 0$$

Therefore the motion is rotational.

Vortex lines are given by the differential equations

$$\frac{dx_1}{\frac{\partial v_3}{\partial x_2}-\frac{\partial v_2}{\partial x_3}}=\frac{dx_2}{\frac{\partial v_1}{\partial x_3}-\frac{\partial v_3}{\partial x_1}}=\frac{dx_3}{\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}}$$

or, $\frac{dx_1}{2c}=\frac{dx_2}{2a}=\frac{dx_3}{2b}$ for fixed time t

Integrating we have $x_1=\frac{c}{a}x_2+C_1$ (C_1 & C_2 being constants of integration)

$$x_1=\frac{c}{b}x_3+C_2$$

Thus, the straight lines $x_1=\frac{c}{a}x_2+C_1=\frac{c}{b}x_3+C_2$ are the vortex lines.

Ex.4.6. Show that the velocity field given by $v_1=\alpha x_2$, $v_2=\alpha x_1$, $v_3=0$ represents rotational motion.

Solution : Curl $\vec{v}=\left(\frac{\partial v_3}{\partial x_2}-\frac{\partial v_2}{\partial x_3}\right)i+\left(\frac{\partial v_1}{\partial x_3}-\frac{\partial v_3}{\partial x_1}\right)j+\left(\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}\right)k$

$$=(0-0)i+(0-0)j+(\alpha+\alpha)k=2\alpha k\neq 0$$

$\therefore$ the motion is rotational.

7.5. INCOMPRESSIBLE FLUID : EQUATION OF CONTINUITY :

In unit-5 we have deduced the equation of continuity from the principle of conservation of mass both in the Lagrangian and the Eulerian methods of description. Also it is shown there that the two forms of the equation of continuity in these two methods of description are equivalent. In the Eulerian method, the equation of continuity is given by

$$\frac{\partial \rho}{\partial t}+\operatorname{div}\left(\rho\vec{v}\right)=0 \qquad \text{......(5.1a)}$$

or, $$\frac{d\rho}{dt}+\rho\operatorname{div.}\vec{v}=0 \qquad \text{......(5.1b)}$$

Now, if any change in the stress applied to a quantity of fluid can not produce any change in the volume of it, that is, it can not be compressed by any amount of stress, then the fluid is called "incompressible" fluid. On the other hand, if change in the volume of the fluid can occur due to the application of an amount of stress then the fluid is said to be "compressible" fluid. Gases are compressible fluids whereas liquids are incompressible. The condition for a fluid to be incompressible is $\frac{d\rho}{dt} = 0$. This follow from (5.1b) since div. $\vec{v} = \nabla . \vec{v}$ represents the time rate of increase of volume per unit volume which must be zero for the incompressible fluid.

7.6. CONSTITUTIVE EQUATIONS : PERFECT FLUID :

We have already stated that a perfect fluid is characterized by the fact that it is incapable of exerting any shearing stress on the adjacent layers in its contact in resisting the shearing movement under a very small shearing force. Therefore, the stress vector exerted by a perfect fluid must be normal to the surface. This normal stress which is always compressive is known as the pressure. If $T_i^{(n)}$ (i = 1, 2, 3) are the components of the stress vector acting across a plane whose normal unit vector is $\vec{n}$, and p is the pressure on that plane, then we have

$$T_i^{(n)} = -p\,n_i \quad (i = 1, 2, 3) \qquad ......(6.1)$$

where n_i (i = 1, 2, 3) are components of $\vec{n}$.

Since the stress tensor T_{ij} is given by

$$T_i^{(n)} = T_{ij}\,n_j,$$

we find that

$$T_{ij}\,n_j = T_i^{(n)} = -pn_i \quad \text{(using (6.1))}$$

$$= -\,p\,\delta_{ij}\,n_j$$

Therefore, $\quad T_{ij} = -\,p\,\delta_{ij} \qquad ......(6.2)$

These equations are the constitutive equations of perfect fluid.

For perfect fluid the existence of compressive normal stress or the pressure leads to the following theorem :

Theorem : The pressure at any point in a perfect fluid has the same magnitude in every direction.

Let dS and dS′ be any two arbitrary surface elements passing through a point P, having normals $\vec{n} = (n_1, n_2, n_3)$ and $\vec{n'} = (n_1', n_2', n_3')$ respectively. If p and p′ are the pressures, that is, the compressive normal stresses at P acting across surface elements dS and dS′ respectively, then we have

$$T_i^{(n)} = pn_i, \quad T_i^{(n')} = -p' \, n_i' \qquad \ldots\ldots(6.3)$$

$$T_{ij} \, n_j = -pn_i, \quad T_{ij} n_j' = -p' n_i' \qquad \ldots\ldots(6.4)$$

where T_{ij} is the stress tensor at the fluid point P, and $T_i^{(n)}$ and $T_i^{(n')}$ are respectively the stress vectors acting at P across the elements of surface dS and dS′ From (6.4)

$$\text{we have} \quad \left.\begin{array}{ll} \text{We have} & T_{ij} \, n_j \, n_i' = -pn_i n_i' \\ \text{and} & T_{ij} \, n_j' \, n_i = -p' n_i' \, n_i \end{array}\right\} \qquad \ldots\ldots(6.5)$$

Now, $T_{ij} n_j' n_i = T_{ji} n_i' n_j$ (interchanging dummy suffixes)

$= T_{ij} n_i' n_j$ (since $T_{ji} = T_{ij}$ because the stress tensor is symmetric)

Therefore, from (6.5) we have

$$pn_i n_i' = p' n_i n_i'$$

or, $\quad (p - p')n_i n_i' = 0$

Since n_i and n_i' represent arbitrary directions, we must have $p - p' = 0$ or, $p = p'$. Hence the theorem.

It should be noted that since the viscous fluid can exert shearing stress when it is in motion the above theorem does not remain valid. In fact, the notion of fluid pressure as the compressive normal stress, that is, in the sense of hydrostatic pressure may not hold good. Still one can define the pressure p at a point in a viscous fluid in motion as the average normal compressive stress at the point, that is, by the following relation :

$$p = -\frac{1}{3}\left(T_{11} + T_{22} + T_{33}\right) = -\frac{1}{3} T_{ii} \qquad \ldots\ldots(6.6)$$

Of course, as the viscous fluid at rest can not exert shear stress the hydrostatic pressure p_0 at a point has the same magnitude in every direction, and the stress tensor is given by $T_{ij} = -p_0 \delta_{ij}$ ......(6.7)

7.7 CONSTITUTIVE EQUATIONS : VISCOUS FLUID

The stress exerted by a viscous fluid is dependent on the rate of change of strain, and the stress vanishes when this strain rate or rate of deformation is zero. For Newtonian viscous fluid or linearly viscous fluid the stress is a linear function of strain rate. Thus, for such linearly viscous fluid in motion one can have

$$T_{ij} = B_{ij} + C_{ijkl}\, d_{kl} \qquad \ldots\ldots(7.1)$$

when $$d_{kl} = \frac{1}{2}(v_{k,l} + v_{l,k}) \qquad \ldots\ldots(7.2)$$

is the rate of strain, $\bar{v} = (v_1, v_2, v_3)$ being the velocity of the fluid.

Now, the equation (7.1) for fluid in motion must be reduced to the equation (6.7) when the fluid is at rest for which case the pressure p as given in (6.6) must be equal to p_0 and the strain rate $d_{kl} = 0$. Therefore, when $p = p_0$,

and $d_{kl} = 0$ we must have (for fluid at rest)

$$T_{ij} = B_{ij} = -p_0\delta_{ij} = -p\delta_{ij}$$

Therefore, the equation (7.1) becomes

$$T_{ij} = -p\delta_{ij} + C_{ijkl}\, d_{kl} \qquad \ldots\ldots(7.3)$$

for linearly viscous fluid in motion. It is to be noted that C_{ijkl} is symmetric with respect to the first two indices as well as last two indices because both T_{ij} and δ_{ij} are symmetric tensors. If we consider an isotropic homogeneous linearly viscous fluid whose physical properties are the same at all points and are identical in respect of every directions from any point, then C_{ijkl} must be an isotropic tensor of order four. This tensor can be written as

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \upsilon\delta_{il}\delta_{jk} \qquad \ldots\ldots(7.4)$$

where λ, μ, υ are constants. Now, since

$C_{ijkl} = C_{ijlk}$ (as it is symmetric w.r. to the last two indices)

we have

$$\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \upsilon\delta_{il}\delta_{jk} = \lambda\delta_{ij}\delta_{lk} + \mu\delta_{il}\delta_{jk} + \upsilon\delta_{ik}\delta_{jl}$$

or, $\mu(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \upsilon(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) = 0 \quad (\because\ \delta_{kl} = \delta_{lk})$

or, $(\mu - \upsilon)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) = 0$

As this equation is true for all values of i, j, k, l, we put

$i = 1, k = 1, j = 2, l = 2$ in this equation to get

$$(\mu - \upsilon)(\delta_{11}\delta_{22} - \delta_{12}\delta_{21}) = 0$$

$$\text{or,} \quad \mu - \upsilon = 0 \quad \text{or,} \ \mu = \upsilon \qquad \text{......(7.5)}$$

Consequently, we have from (7.4)

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \qquad \text{......(7.6)}$$

Using (7.6) we have from (7.3)

$$T_{ij} = -p\delta_{ij} + \lambda\delta_{ij}\delta_{kl}d_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})d_{kl}$$

$$\text{or,} \quad T_{ij} = -p\delta_{ij} + \lambda D\delta_{ij} + 2\mu d_{ij} \qquad \text{......(7.7)}$$

$$\text{where} \quad D = d_{ll} = v_{l,l} = \text{div } \bar{v} = \nabla . \bar{v} \qquad \text{......(7.8)}$$

These are the constitutive equations for isotropic homogeneous linearly viscous fluid. The constants λ and μ are the viscosity coefficients. For incompressible fluid we have $\frac{\partial\rho}{\partial t} = 0$ or, div $\bar{v} = 0$, and therefore, the constitutive equations for isotropic homogeneous linearly viscous incompressible fluid are given by

$$T_{ij} = -p\delta_{ij} + 2\mu d_{ij} \qquad \text{......(7.9)}$$

Again, from (7.7) we have

$$T_{ii} = -p\delta_{ii} + \lambda D\delta_{ii} + 2\mu d_{ii}$$

$$= -3p + 3\lambda D + 2\mu D \quad \text{(using (7.8))}$$

or, since $T_{ii} = -3p$ (from (6.6)), we have

$$-3p = -3p + (3\lambda + 2\mu)D$$

$$\text{or,} \quad (3\lambda + 2\mu)D = 0$$

For compressible fluid $D \neq 0$, then we have

$$3\lambda + 2\lambda = 0 \quad \text{or,} \ \lambda = -\frac{2\mu}{3} \qquad \text{......(7.10)}$$

Using (7.10) we have from (7.7)

$$T_{ij} = -p\delta_{ij} - \frac{2\mu}{3}D\delta_{ij} + 2\mu d_{ij}$$

$$= -\left(p + \frac{2\mu}{3}D\right)\delta_{ij} + 2\mu d_{ij} \qquad \text{......(7.11)}$$

These are the constitutive equations for isotropic homogeneous linearly viscous compressible fluid.

Unit : 8 ☐ Equations of Motion of Fluid

8.1. EULER'S DYNAMICAL EQUATIONS OF MOTION

In unit-5, we have deduced the equations of motion of the continuum from the principle of balance of linear momentum, which states that the time rate of change of total linear momentum of any specific portion of the continuum is equal to the resultant external force acting on this portion.

These equations are

$$T_{ij,j} + \rho F_i = \rho \frac{dv_i}{dt} \tag{1.1}$$

where T_{ij} is the stress tensor, F_i the body force per unit mass, and v_i the velocity of the continuum.

Now, for perfect fluid we use the constitutive equation (6.2) of unit-7, and substitute this expression for the stress tensor into (1.1) to obtain

$$\left(-p\delta_{ij}\right)_{,j} + \rho F_i = \rho \frac{dv_i}{dt}$$

or, $$-p_{,j}\,\delta_{ij} + \rho F_i = \rho \frac{dv_i}{dt}$$

or, $$\frac{dv_i}{dt} \equiv \frac{\partial v_i}{\partial t} + \left(\vec{v}.\nabla\right)v_i = F_i - \frac{1}{\rho}p_{,i} \tag{1.2}$$

$$(i = 1, 2, 3)$$

These are Euler's equations of motion for a perfect fluid in Eulerian method. These equations can be written in the following form in the vector notation :

$$\frac{d\vec{v}}{dt} \equiv \frac{\partial \vec{v}}{\partial t} + \left(\vec{v}.\nabla\right)\vec{v} = \vec{F} - \frac{1}{\rho}\nabla p \tag{1.3}$$

$$(\nabla p = \text{grad } p)$$

8.2. LAGRANGE'S EQUATIONS OF MOTION

We can obtain the equations of motion in Lagrangian method by following the motion of the fluid particle (the material point) identified by its initial coordinates (X_1, X_2, X_3) at time $t = 0$. As this particle occupies a position (x_1, x_2, x_3) at time t, the

acceleration components of this fluid particle at that time t are $\frac{\partial^2 x_i}{\partial t^2}$ (i = 1, 2, 3).
Consequently, the equations of motion (1.2) become

$$\frac{\partial^2 x_i}{\partial t^2} = F_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} \tag{2.1}$$

$$(i = 1, 2, 3)$$

Let us now convert the partial derivatives with respect to x_i (i = 1, 2, 3) into those with respect to the independent variables X_i = (i = 1, 2, 3) of the Lagrangian method. For this, we multiply equations (2.1) by $\frac{\partial x_i}{\partial X_k}$ and sum over i to obtain

$$\frac{\partial^2 x_i}{\partial t^2}\frac{\partial x_i}{\partial X_k} = F_i\frac{\partial x_i}{\partial X_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i}\frac{\partial x_i}{\partial X_k}$$

or,
$$\left(F_i - \frac{\partial^2 x_i}{\partial t^2}\right)\frac{\partial x_i}{\partial X_k} = \frac{1}{\rho}\frac{\partial p}{\partial X_k} \tag{2.2}$$

$$(k = 1, 2, 3)$$

These are the equations of motion of perfect fluid in Lagrangian method of description.

8.3. INTEGRALS OF EULER'S EQUATIONS OF MOTION

We have four equations, three of which are the Euler's equations of motion (1.2) and the other is the equation of continuity

$$\frac{1}{\rho}\frac{d\rho}{dt} + \text{div.}\,\vec{v} = 0 \tag{3.1}$$

for determination of five unknown quantities v_i (i = 1, 2, 3), p and ρ which are functions of the independent variables x_i = (i = 1, 2, 3) and t. These five unknown quantities may be reduced to four unknown quantities if one assumes a relationship between the pressure p and the density ρ . Assuming such a relationship between p and ρ so that $\int \frac{dp}{\rho}$ exists, we introduce a pressure potential P given as

$$\left.\begin{aligned} P &= \int \frac{dp}{\rho} \\ \text{or,}\quad dP &= \frac{dp}{\rho} \end{aligned}\right\} \tag{3.2}$$

From this, it follows that

$$\frac{\partial P}{\partial x_i} dx_i + \frac{\partial P}{\partial t} dt = \frac{1}{\rho}\left(\frac{\partial p}{\partial x_i} dx_i + \frac{\partial p}{\partial t} dt\right) \quad \left(\begin{array}{l}\text{since P and p are} \\ \text{functions of } x_i \text{ and } t\end{array}\right)$$

Therefore, $\dfrac{\partial P}{\partial x_i} = \dfrac{1}{\rho}\dfrac{\partial p}{\partial x_i}$ (equating the coefficients of dx_i)

or, $\text{grad } P = \dfrac{1}{\rho} \text{ grad } p$

or, $$\frac{1}{\rho} \text{ grad } p = \text{grad}\left(\int \frac{dp}{\rho}\right) \tag{3.3}$$

For conservative external body forces which are derivable from a potential U (say), that is

$$\vec{F} = -\text{grad } U \tag{3.4}$$

we have from Euler's equations (1.3) and (3.3)

$$\frac{\partial \bar{v}}{\partial t} + (\bar{v}.\nabla)\bar{v} = -\text{grad } U - \frac{1}{\rho} \text{ grad } p = -\text{grad } U - \text{grad}\left(\int \frac{dp}{\rho}\right)$$

$$= -\text{grad}\left(U + \int \frac{dp}{\rho}\right) \tag{3.5}$$

Now, we use the following vector identity

$$\bar{v} \times \text{curl}\bar{v} = \frac{1}{2} \text{grad}\left(\bar{v}^2\right) - (\bar{v}.\nabla)\bar{v}$$

or, $$(\bar{v}.\nabla)\bar{v} = \frac{1}{2} \text{grad}\left(\bar{v}^2\right) - \bar{v} \times \text{curl } \bar{v}$$

Substituting the expression for $(\bar{v}.\nabla)\,\bar{v}$ from this identity into (3.5), we have

$$\frac{\partial \bar{v}}{\partial t} - \bar{v} \times \text{curl } \bar{v} = -\text{ grad}\left(U + \int \frac{dp}{\rho}\right) - \frac{1}{2} \text{ grad}\left(\bar{v}^2\right)$$

$$= -\text{grad}\left(\frac{\bar{v}^2}{2} + U + \int \frac{dp}{\rho}\right) = -\text{grad } H \tag{3.6}$$

where $$H = \frac{1}{2}\bar{v}^2 + U + \int \frac{dp}{\rho} \tag{3.7}$$

which represents the total specific energy being the sum of kinetic, potential and pressure energies per unit mass. The equation (3.6) is the Euler's equation of motion when the body forces are conservative and the pressure is function of density alone.

Now, we shall discuss few cases for which the Euler's equation (3.6) have integrals.

Case I. If the fluid motion is irrotational, that is, curl $\vec{v} = 0$ at every point of the continuum, then the velocity potential ϕ exists, that is,

$$\vec{v} = -\text{ grad } \phi$$

Consequently, we have from (3.6)

$$-\frac{\partial}{\partial t}(\text{grad } \phi) = -\text{grad}\left(\frac{\partial \phi}{\partial t}\right) = -\text{grad H}$$

or, $\text{grad}\left(\frac{\partial \phi}{\partial t} - H\right) = 0$

Integrating we have

$$\frac{\partial \phi}{\partial t} - H = \text{a function of t only} = F(t) \text{ (say)}$$

or, $\frac{\partial \phi}{\partial t} - U - \frac{\vec{v}^2}{2} - \int \frac{dp}{\rho} = F(t)$ (3.8)

Thus, (3.8) is the integral of Euler's equation of motion for this case of irrotational flow of fluid. This integral is the Generalised Bernoulli's equation or pressure equation.

Case II. Fluid motion is supposed to be rotational but steady. For this

we have $\frac{\partial \vec{v}}{\partial t} = 0$ or, $\vec{v} = \vec{v}(x_1, x_2, x_3)$ (3.9)

Then, we have from (3.6)

$\vec{v} \times \text{curl } \vec{v} = \text{grad H}$ (3.10)

or, $\vec{v}.(\vec{v} \times \text{curl } \vec{v}) = \vec{v}.\text{ grad H}$ (3.11)

Since the vector $\vec{v}$ is perpendicular to $\vec{v} \times \text{curl } \vec{v}$ we have the L.H.S of (3.11) is equal to zero, and therefore

$$\vec{v}.\text{ grad H} = 0$$

That is, grad H is perpendicular to $\vec{v}$. For steady motion $\vec{v}$ is tangential to the stream line, and consequently the vector grad H must be normal to the stream line. Therefore, the component of grad H along the stream line is zero. That is,

$$\frac{\partial H}{\partial s} = 0 \tag{3.12}$$

where ds represents the element of arc of the stream line.

Integrating we have

$$H = \frac{\vec{v}^2}{2} + U + \int \frac{dp}{\rho} = \text{constant} = C \text{ (say)} \tag{3.13}$$

where the "constant" C remains constant at every point of the stream line but it differs, in general, from one stream line to another. (3.13) is the integral of Euler's equation of motion for steady rotational flow. This is known as Bernoulli's equation along a stream line.

For uniform steady motion $\vec{v}$ = constant, that us, curl $\vec{v} = 0$, and we have

grad H = 0 or, H = constant independent of x_i and t.

$$\text{or,}\quad U + \int \frac{dp}{\rho} = \text{constant} - \frac{\vec{v}^2}{2} = \text{constant} \tag{3.14}$$

Again, from (3.10) we get

$$\text{curl } \vec{v}.(\vec{v} \times \text{curl } \vec{v}) = \text{curl } \vec{v}.\text{ grad H}$$

or, curl $\vec{v}$. grad H = 0 (since curl $\vec{v}$ is perpendiculat to the vector $\vec{v} \times$ curl $\vec{v}$)

Therefore, grad H is perpendicular to the vector curl $\vec{v}$. Now, as the vorticity vector curl $\vec{v}$ is tangential to the vortex line, grad H must be normal to the vortex line, and consequently its component along the vortex line will be zero. Hence,

$$\frac{\partial H}{\partial s'} = 0$$

where ds′ is the arc element of the vortex line. Thus, on integration,

$$H = \frac{\vec{v}^2}{2} + U + \int \frac{dp}{\rho} = \text{constant} = C' \text{ (say)} \tag{3.15}$$

where C′ remains constant at every point along the vortex line but it differs, in general from one vortex line to another. The equation (3.15) is the integral of Euler's equation of motion for steady rotational flow, and is called Bernoulli's equation along a vortex line.

Since H remains constant both along a stream line and a vortex line, it must remain constant over a surface containing both these lines. This surface is known as Lamb surface.

Case III. Let us now suppose that the fluid motion is steady and irrotational. Thus, we have $\frac{\partial \vec{v}}{\partial t} = 0$ and curl $\vec{v} = 0$.

Then, from (3.6) we have grad $H = 0$, and on integration we get

$$H = \frac{1}{2}\vec{v}^2 + U + \int \frac{dp}{\rho} = \text{constant independent of } x_i \text{ and } t.$$

$$= C'' \text{ (say)} \qquad (3.16)$$

C″ remains constant throughout the fluid flow at all times. This integral (3.16) of the Euler's equation of motion is the Bernoulli-Euler integral or Bernoulli's equation for steady irrotational flow.

Exercise 3.1. For the steady fluid motion if the stream lines and vortex lines are parallel show that the sum of potential energy, kinetic energy and pressure energy per unit mass is an absolute constant.

Hints. For steady motion $\frac{\partial \vec{v}}{\partial t} = 0$. If stream lines and vortex lines are parallel, $\vec{v} \times \text{curl } \vec{v} = 0$. Then use (3.6).

Exercise 3.2. For homogeneous incompressible fluid moving steadily under the action of gravity only, find the Bernoulli's equation along a stream line.

Hints For incompressible fluid $\frac{d\rho}{dt} = 0$. The potential U is given by $-\frac{\partial U}{\partial x_3} = -g$ if x_3 axis is directed vertically upwards.

or, $U = gx_3$

Then use (3.6) and $\int \frac{dp}{\rho} = \frac{p}{\rho}$ (as ρ = constant)

[Ans : $x_3 + \frac{p}{\rho g} + \frac{\bar{v}^2}{2g}$ = constant along a stream line]

8.4. KELVINE'S THEOREM ON MINIMUM KINETIC ENERGY

Kelvine's theorem on minimum kinetic energy states that the irrotational motion of an incompressible perfect fluid occupying a simply connected regioun has less kinetic energy than any other motion of the fluid having on its boundary the same normal velocity as in the irrotational motion.

Thus, the irrotational motion has the minimum kinetic energy.

Proof of the theorem :

Let us suppose that an incompressible perfect fluid is occupying a region V bounded by a closed surface Σ. Let ρ be the density of the fluid and T the kinetic energy of the fluid moving irrotationally. Let $\bar{v}$ (v_1, v_2, v_3) be the fluid velocity. Then we have

$$T = \frac{1}{2}\rho \int_V \bar{v}^2 \, dV \tag{4.1}$$

Since the fluid motion is irrotational, a velocity potential ϕ exists such that

$$v_i = -\frac{\partial \phi}{\partial x_i} \ (i = 1, 2, 3) \text{ at every point in the region V.} \tag{4.2}$$

Also, since the fluid is incompressible we have div $\bar{v} = 0$ (4.3)

at every interior point.

Now, let T′ be the kinetic energy of any other possible state of fluid motion, and $\bar{v}' = (v_1', v_2', v_3')$ is the corresponding fluid velocity. Then we have

$$T' = \frac{1}{2}\rho \int_V \bar{v}'^2 \, dV \tag{4.4}$$

The velocity $\vec{v}'$ must satisfy the equation of continuity for incompressible fluid, that is,

$$\operatorname{div} \vec{v}' = 0 \tag{4.5}$$

Now, since these two motions have the same normal velocity on Σ then we must have

$$\vec{n}.\vec{v} = \vec{n}.\vec{v}', \quad \text{or } \vec{n}(\vec{v} - \vec{v}') = 0 \tag{4.6}$$

where $\vec{n} = (n_1, n_2, n_3)$ is unit normal vector at a point on Σ. From (4.1) and (4.4) we have

$$T' - T = \frac{1}{2}\rho \int_V \left(\vec{v}'^2 - \vec{v}^2\right) dV$$

$$= \frac{1}{2}\rho \int_V \left\{\left(v_1'^2 - v_1^2\right) + \left(v_2'^2 - v_2^2\right) + \left(v_3'^2 - v_3^2\right)\right\} dV$$

$$= \frac{1}{2}\rho \int_V \left\{\begin{matrix}\left(v_1' - v_1\right)^2 + 2v_1\left(v_1' - v_1\right) + \left(v_2' - v_2\right)^2 + 2v_2\left(v_2' - v_2\right) \\ + \left(v_3' - v_3\right)^2 + 2v_3\left(v_3' - v_3\right)\end{matrix}\right\} dV$$

$$= \frac{1}{2}\rho \int_V \left\{\left(v_1' - v_1\right)^2 + \left(v_2' - v_2\right)^2 + \left(v_3' - v_3\right)^2\right\} dV$$

$$+ \rho \int_V \left\{v_1\left(v_1' - v_1\right) + v_2\left(v_2' - v_2\right) + v_3\left(v_3' - v_3\right)\right\} dV$$

$$= I_1 + I_2 \text{ (say)} \tag{4.7}$$

where $I_1 = \frac{1}{2}\rho \int_V \left\{\left(v_1' - v_1\right)^2 + \left(v_2' - v_2\right)^2 + \left(v_3' - v_3\right)^2\right\} dV$

$$= \frac{1}{2}\rho \int_V \left(\vec{v}' - \vec{v}\right)^2 dV \tag{4.8}$$

and $I_2 = \rho \int_V \left\{v_1\left(v_1' - v_1\right) + v_2\left(v_2' - v_2\right) + v_3\left(v_3' - v_3\right)\right\} dV$

Now, $I_2 = \rho \int_V v_i\left(v_i' - v_i\right) dV = -\rho \int_V \frac{\partial \phi}{\partial x_i}\left(v_i' - v_i\right) dV$ [using (4.2)]

$$= -\rho \int_V \left\{ \frac{\partial}{\partial x_i} \left[\phi (v_i' - v_i) \right] - \phi \frac{\partial}{\partial x_i} (v_i' - v_i) \right\} dV$$

$$= -\rho \int_V \operatorname{div} \left\{ \phi \left(\vec{v}' - \vec{v} \right) \right\} dV + \rho \int_V \phi \operatorname{div} \left(\vec{v}' - \vec{v} \right) dV$$

$$= -\rho \int_\Sigma \phi . \vec{n} . \left(\vec{v}' - \vec{v} \right) d\Sigma + \rho \int_V \phi \left\{ \operatorname{div} \vec{v}' - \operatorname{div} \vec{v} \right\} dV \qquad (4.9)$$

(using Gauss's divergence theorem for the first integral)

From (4.6), it is found that the surface integral on the R.H.S. of (4.9) vanishes. Also, due to the conditions in (4.3) and (4.5), the second term, that is, the volume integral on the R.H.S. vanishes. Therefore, we have $I_2 = 0$, and hence

$$T' - T = I_1 = \frac{1}{2} \rho \int_V \left(\vec{v}' - \vec{v} \right)^2 dV = \text{a positive quantity}$$

Consequently, we have $T' - T > 0$

or, $T < T'$

Hence the theorem.

8.5. CONSTANCY OF CIRCULATION

Let us consider the circulation Γ round a closed curve (closed circuit) C in the fluid flow. Let $\vec{v} = (v_1, v_2, v_3)$ be the velocity., The circulation Γ is given as

$$\Gamma = \oint_C \vec{v} . d\vec{r} = \int_C v_i \, dx_i \qquad (5.1)$$

where $d\vec{r} = (dx_1, dx_2, dx_3)$ is the directed element of the curve.

Therefore, $\frac{d\Gamma}{dt} = \oint_C \frac{d}{dt} (v_i \, dx_i)$ (since the region of the integration is a definite portion of the fluid, that is , on the circuit C)

or, $$\frac{d\Gamma}{dt} = \oint_C \frac{dv_i}{dt} dx_i + \oint_C v_i \frac{d}{dt} (dx_i) \qquad (5.2)$$

Now, $\frac{d}{dt}(dx_i) = dv_i$ (since $\frac{d}{dt}(dx_i)$ represents the time rate of the relative position of a fluid particle w.r. to its neighbour, therefore, it is the relative velocity of the fluid particle w.r. to its neighbour)

$$\text{or,} \quad \frac{d}{dt}(dx_i) = \frac{\partial v_i}{\partial x_j} dx_j$$

Using this, we have, from (5.2),

$$\frac{d\Gamma}{dt} = \oint_C \frac{dv_i}{dt} dx_i + \oint_C v_i \frac{\partial v_i}{\partial x_j} dx_j \qquad (5.3)$$

Now, we use Euler's equation of motion for perfect fluid, given by

$$\frac{dv_i}{dt} = F_i - \frac{1}{\rho} p_{,i}$$

For the conservative body force, that is, if F_i are given as $F_i = -\frac{\partial U}{\partial x_i} = -U_{,i}$, where U is a potential function, we have the Euler's equation for incompressible perfect fluid as

$$\frac{dv_i}{dt} = -U_{,i} - \frac{1}{\rho} p_{,i} = -\left(U + \frac{p}{\rho}\right)_{,i}$$

Consequently, the equation (5.3) becomes

$$\frac{d\Gamma}{dt} = -\oint_C \left(U + \frac{p}{\rho}\right)_{,i} dx_i + \oint_C v_i \frac{\partial v_i}{\partial x_j} dx_j$$

$$= -\oint_C d\left(U + \frac{p}{\rho}\right) + \frac{1}{2}\oint \frac{\partial}{\partial x_j}\left(v_i^2\right) dx_j$$

$$= -\oint_C d\left(U + \frac{p}{\rho}\right) + \frac{1}{2}\oint_C d\left(v_i^2\right)$$

$$= \oint_C d\left(\frac{1}{2} v_i^2 - U - \frac{p}{\rho}\right) = 0$$

since C is a closed curve and $\frac{1}{2} v_i^2 - V - \frac{p}{\rho}$ is a single-valued function.

Therefore,

$$\Gamma = \text{constant}$$

Thus, the circulation around any closed curve of fluid particles moving along with the fluid remains constant throughout the motion if the external body forces acting on a perfect incompressible fluid are conservative. This is Kelvine's theorem on the constancy of circulation.

8.6. MOTION IN TWO DIMENSIONS : SOURCES, SINKS AND DOUBLETS

If the velocity $\vec{v} = (v_1, v_2, v_3)$ of the fluid motion is such that $v_1 \equiv v_1(x_1, x_2, t)$, $v_2 \equiv v_2(x_1, x_2, t)$, $v_3 = 0$, then the motion takes place in the layers parallel to x_1x_2-plane, that is, the motion is the same in each of these plane layers. Such fluid motion is the motion in two dimensions.

For the two-dimensional motion the differential equation of the stream line is given by

$$\frac{dx_1}{v_1(x_1, x_2, t)} = \frac{dx_2}{v_2(x_1, x_2, t)} \qquad (6.1)$$

or, $\quad v_2 dx_1 - v_1 dx_2 = 0.$

The L.H.S. of this equation will be a perfect differential if a function ψ exists such that

$$v_1 = -\frac{\partial \psi}{\partial x_2}, \quad v_2 = \frac{\partial \psi}{\partial x_1} \qquad (6.2)$$

Therefore, $\quad d\psi = \frac{d\psi}{\partial x_1} dx_1 + \frac{d\psi}{dx_2} dx_2 = 0$

$$\text{or, } \psi = \text{constant} \qquad (6.3)$$

(6.3) represents the stream line ; and ψ is the stream function.

The vorticity vector $\vec{w} = \text{curl } \vec{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & 0 \end{vmatrix}$

or, $\quad \vec{w} = k\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)$

$$= k\left(\frac{\partial^2 \psi}{\partial x_1{}^2} + \frac{\partial^2 \psi}{\partial x_2^2}\right) \quad \text{(using (6.2))}$$

Therefore, for irrotational motion we have

$$\frac{\partial^2\psi}{\partial x_1^2}+\frac{\partial^2\psi}{\partial x_2^2}=0 \qquad (6.4)$$

Also, for an irrotational motion a velocity potential ϕ exists such that

$$v_1=-\frac{\partial\phi}{\partial x_1},\quad v_2=-\frac{\partial\phi}{\partial x_2} \qquad (6.5)$$

For incompressible fluid we have $\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}=0$, $\left(\because \text{ div } \vec{v}=0\right)$ and consequently from (6.5) it follows that

$$\frac{\partial^2\phi}{\partial x_1^2}+\frac{\partial^2\phi}{\partial x_2^2}=0 \qquad (6.6)$$

The curve ϕ = constant is the curve of equal velocity potential. From (6.2) and (6.5) it follows that $\frac{\partial\phi}{\partial x_1}=\frac{\partial\psi}{\partial x_2}$ and $\frac{\partial\phi}{\partial x_2}=-\frac{\partial\psi}{\partial x_1}$ (6.7)

These relations remain unaltered if one writes $-\psi$ for ϕ, and ϕ for ψ Consequently, if we take the curves ψ = constant as the equipotential curves, and ϕ = constant as the stream line we have another possible case of irrotional motion.

Now, the fundamental solution of (6.6) is of the form : $\phi = C \log r$ where r is the distance from a fixed point (say, the origin), that is $r^2 = x_1^2 + x_2^2$. This is the case of two-dimensional source if $C=-\frac{m}{2\pi}$,

$$\text{or,}\quad \phi=-\frac{m}{2\pi}\log r \qquad (6.8)$$

and if $C=\frac{m}{2\pi}$ or, $\phi=\frac{m}{2\pi}\log r$ (6.9)

it represents a two-dimensional sink. The constant m is the 'strength' of the source as it is the outward flux across a circle surrounding the point (soruce), that is,

$$-\frac{\partial\phi}{\partial r}\cdot 2\pi r=m \qquad (6.10)$$

The source is, thus, a point from which the fluid flows out uniformly in all direction. Obviously, a sink can be looked as a negative source. The existence of a source or a sink implies a continual creation or annihilation of fluid at the point under consideration.

A combination of the equal and opposite sources that is, a source and a sink of strengths m and -m respectively, situated at a distance δS apart such that in the limit $\delta S \to 0$ and $m \to \infty$, the product $m\delta S$ remains finite and is equal to μ (say), is called a double source or doublet. μ is the strength, and the line δS drawn from the sink of strengh -m to the source of strength +m is called the axis of the doublet.

Now, the velocity potential at P for a doublet with "sink" -m and "source" m situated at A and B, that is, the axis being $AB = \delta s$, can be found by the addition of the velocity potentials for the sink and source (see fig. 6.1)

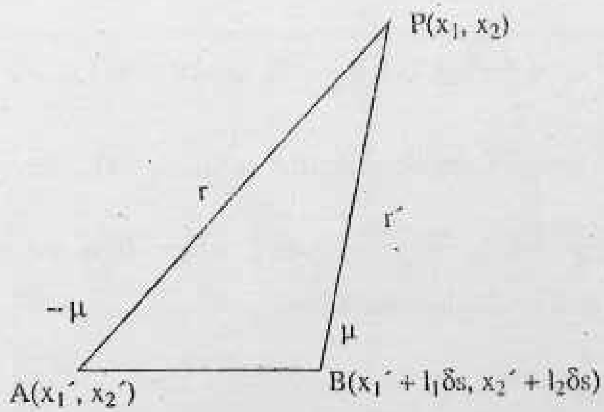


Fig. 6.1

Let (x_1, x_2) be the co-ordinates of P. If l_1, l_2 are the direction cosines of the axis δs of the doublet, then the coordinates of A and B can be taken as (x_1', x_2') and $(x_1' + l_1\delta s, x_2' + l_2\delta s)$ respectively. Then the velocity potential for the doublet at the point P is given by

$$\phi = \frac{m}{2\pi}\log r - \frac{m}{2\pi}\log r' = -\frac{m}{2\pi}(\log r' - \log r) \;\ldots\ldots\; (6.11)$$

where $r = AP$ and $r' = BP$.

Let $\log r = f(x_1', x_2')$, then $\log r' = f(x_1' + \ell_1\delta s, x_2' + \ell_2\delta s)$

Then, $\log r' - \log r = f(x_1' + \ell_1\delta s,\ x_2' + \ell_2\delta s) - f(x_1', x_2')$

$$= \left(\ell_1 \frac{\partial}{\partial x_1'} + \ell_2 \frac{\partial}{\partial x_2'}\right) f(x_1',\ x_2')\delta s$$

$$= \left\{\left(\ell_1 \frac{\partial}{\partial x_1'} + \ell_2 \frac{\partial}{\partial x_2'}\right) \log r\right\}(\delta s)$$

Therefore, $\phi = -\frac{m\delta s}{2\pi}\left(\ell_1 \frac{\partial}{\partial x_1'} + \ell_2 \frac{\partial}{\partial x_2'}\right)\log r$

Now, since $m\delta s = \mu$ in the limit $m \to \infty$, $\delta s \to 0$, μ being the strength of the doublet, we have

$$\phi = -\frac{\mu}{2\pi}\left\{\frac{\ell_1}{r}\frac{\partial r}{\partial x_1'} + \frac{\ell_2}{r}\frac{\partial r}{\partial x_2'}\right\}$$

$$= +\frac{\mu}{2\pi r}\left\{\ell_1 \frac{x_1 - x_1'}{r} + \ell_2 \frac{x_2 - x_2'}{r}\right\} \text{ (since } r^2 = (x_1 - x'_1)^2 + (x_2 - x_2')^2$$

Since $\frac{x_1 - x_1'}{r}$, $\frac{x_2 - x_2'}{r}$ are the direction cosines of AP, then

$\ell_1 \frac{x_1 - x_1'}{r} + \ell_2 \frac{x_2 - x_2'}{r} = \cos\hat{\theta}$ where $\hat{\theta}$ is the angle between the increasing r and the axis of the doublet. Then

$$\phi = \frac{\mu}{2\pi r}\cos\hat{\theta} \qquad \ldots\ldots \quad (6.12)$$

Example 6.1 Find the stream function for a two-dimensional source given by the velocity potential $\phi = -\frac{m}{2\pi}\log r$ where r is the distance from the source point.

Solution : $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ if we take the source is at the origin. Now,

$$v_1 = -\frac{\partial\phi}{\partial x_1} = -\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x_1}$$

or, $v_1 = \frac{m}{2\pi r}\frac{x_1}{r}$ (Since $r^2 = x_1^2 + x_2^2$)

Similarly, $v_2 = = \frac{mx_2}{2\pi r^2}$

Stream line is given by $\frac{dx_1}{v_1} = \frac{dx_2}{v_2}$ or, $\frac{dx_1}{x_1} = \frac{dx_2}{x_2}$

Integrating we have $\log x_1 = \log x_2 + \text{constant}$

or, $\frac{x_2}{x_1} = \text{constant}$ or, $\frac{r \sin\theta}{r\cos\theta} = \text{constant}$

or, $\tan\theta = \text{constant}$

Therefore, $\theta = \text{constant}$ is the stream line. Consequently, stream function $\psi = C\theta$ where C is a constant. Now, since $v_1 = -\frac{\partial\psi}{\partial x_2}$, $v_2 = \frac{\partial\psi}{\partial x_1}$ we have

$$v_1 = \frac{mx_1}{2\pi r^2} = -\frac{\partial}{\partial x_2}(C\theta) = -C\frac{\partial}{\partial x_2}\left(\tan^{-1}\frac{x_2}{x_1}\right) \quad \left[\because \tan\theta = \frac{x_2}{x_1}\right]$$

$$= -C\frac{\frac{1}{x_1}}{1+\left(\frac{x_2}{x_1}\right)^2} = -C\frac{x_1}{x_1^2+x_2^2} = -C\frac{x_1}{r^2}$$

$$\therefore \ C = -\frac{m}{2\pi}$$

Consequently, the stream function ψ is given by

$$\psi = -\frac{m}{2\pi}\theta \qquad \ldots\ldots \ (6.13)$$

Now, we have seen above that if we write $-\psi$ for ϕ, and ϕ for ψ, then ϕ = constant and ψ = constant represent, repectively, the stream line and the equipotential curve for another possible fluid motion. Therefore, in this case of fluid motion, the velocity potential ϕ' and stream function ψ' are, respectively, given by

$$\left.\begin{aligned} \phi' &= -\frac{m}{2\pi}\theta \\ \psi' &= +\frac{m}{2\pi}\log r \end{aligned}\right\} \qquad (6.14)$$

This is the case of simple vortex. The velocity components are $v_\theta = -\frac{1}{r}\frac{\partial\phi'}{\partial\theta} = \frac{m}{2\pi r}$, $v_r = -\frac{\partial\phi'}{\partial r} = 0$.

Therefore, $\Gamma = \oint \vec{v}.d\vec{s} = \oint_0^{2\pi} v_\theta \, r \, d\theta = \oint_0^{2\pi} \frac{mr}{2\pi r} d\theta = m$

Thus, m, here, represents the circulation round the vortex.

8.7. VISCOUS FLOW : NAVIER STOKES EQUATION

In unit – 7, section 7.7, we have found the constitutive equations for isotropic homogeneous linearly viscous fluid as

$$T_{ij} = -\left(p + \frac{2\mu}{3} D\right)\delta_{ij} + 2\mu \, d_{ij} \qquad (7.1)$$

where $D = \text{div } \vec{v} \equiv \nabla . \vec{v}$ (7.2)

and $d_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right)$ (7.3)

Here, μ is the coefficient of viscosity.

Also, we have the equation of motion of continuum as

$$T_{ij,j} + \rho F_i = \rho \frac{dv_i}{dt} \qquad (7.4)$$

From (7.1), we get

$$T_{ij,j} = -p_{,j}\delta_{ij} - \frac{2\mu}{3}\delta_{ij}D_{,j} + \mu\left(v_{i,jj} + v_{j,ij}\right) \quad \text{(using (7.3))}$$

or, $$T_{ij,j} = -p_{,i} - \frac{2\mu}{3}D_{,i} + \mu\nabla^2 v_i + \mu\left(v_{j,j}\right)_{,i}$$

$$= -p_{,i} - \frac{2\mu}{3}D_{,i} + \mu\nabla^2 v_i + \mu D_{,i} \quad (\text{since } v_{j,j} = \text{div } \vec{v} = D)$$

or, $$T_{ij,j} = -p_{,i} + \frac{\mu}{3}D_{,i} + \mu\nabla^2 v_i \qquad (7.5)$$

Now, for incompressible fluid D = 0, and therefore, we have

$$T_{ij,j} = -p_{,i} + \mu\nabla^2 v_i \qquad (7.6)$$

Substituting (7.6) into (7.4) we obtain

$$-\frac{\partial p}{\partial x_i} + \mu\nabla^2 v_i + \rho F_i = \rho \frac{dv_i}{dt}$$

or, $\frac{dv_i}{dt} = F_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\nabla^2 v_i$ (7.7)

where $\nu = \frac{\mu}{\rho}$ is the kinematical coefficient of viscosity.

The equation (7.7) can be written in the vector notation as

$$\left.\begin{aligned} \frac{d\bar{v}}{dt} &= \vec{F} - \frac{1}{\rho}\,\text{grad }\ p + \nu\nabla^2\bar{v} \\ \text{or, } \frac{\partial\bar{v}}{\partial t} + (\bar{v}.\nabla)\bar{v} &= \vec{F} - \frac{1}{\rho}\,\text{grad }\ p + \nu\nabla^2\bar{v} \end{aligned}\right\} \quad (7.8)$$

This is Navier Stokes equation of motion for incompressible viscous fluid.

For compressible viscous fluid, we have from (7.4) and (7.5)

$$\rho\frac{dv_i}{dt} = \rho F_i - \frac{\partial p}{\partial x_i} + \frac{\mu}{3}D_{,i} + \mu\nabla^2 v_i$$

or, $\frac{dv_i}{dt} = F_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\nu}{3}D_{,i} + \nu\nabla^2 v_i$ (7.9)

or, in vector notation

$$\frac{\partial\vec{v}}{\partial t} + \left(\vec{v}.\nabla\right)\vec{v} = \vec{F} - \frac{1}{\rho}\,\text{grad}\, p + \frac{\nu}{3}\,\text{grad}(\text{div}\,\vec{v}) + \nu\nabla^2\vec{v}. \quad (7.10)$$

This equation is Navier Stokes equation of motion for compressible viscous fluid.

8.8. CIRCULATION IN VISCOUS FLOW

The circulation Γ round a closed circuit C in a moving viscous fluid with velocity $\vec{v} = (v_1, v_2, v_3)$ is given by

$$\Gamma = \oint_C \vec{v}.d\vec{r} = \oint_C v_i dx_i \quad (8.1)$$

$$\therefore \quad \frac{d\Gamma}{dt} = \frac{d}{dt}\left(\oint_C v_i dx_i\right) = \oint_C \frac{d}{dt}\left(v_i dx_i\right)$$

(since C is a definite portion of the fluid)

$$= \oint_C \frac{dv_i}{dt} dx_i + \oint_C v_i \frac{d}{dt}(dx_i) = \oint_C \frac{dv_i}{dt} dx_i + \oint_C v_i dv_i \quad [\because \frac{d}{dt}(dx_i) = dv_i]$$

$$= \oint_C \frac{dv_i}{dt} dx_i + \oint_C d\left(\frac{1}{2} v_i v_i\right) \qquad (8.2)$$

Now, for conservative body forces, given by $F_i = -\frac{\partial U}{\partial x_i}$ where U is the potential, the Novier Stokes equation of motion (7.9) becomes.

$$\frac{dv_i}{dt} = -\frac{\partial U}{\partial x_i} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\nu}{3} D_{,i} + \nu\nabla^2 v_i \quad (8.3)$$

If the pressure p is a function of density ρ alone, then the pressure potential P is found to be given as

$$dP = \frac{dp}{\rho} \quad \text{or,} \frac{\partial P}{\partial x_i}.dx_i = \frac{1}{\rho}\frac{\partial p}{\partial x_i}.dx_i$$

That is, $\frac{\partial P}{\partial x_i} = \frac{1}{\rho}\frac{\partial p}{\partial x_i}$

Therefore, from (8.3) we have

$$\frac{dv_i}{dt} = \frac{\partial}{\partial x_i}\left\{\frac{\nu}{3} D - U - P\right\} + \nu\nabla^2 v_i \qquad (8.4)$$

Then, from (8.2) and (8.4) we get

$$\frac{d\Gamma}{dt} = \oint_C \frac{\partial}{\partial x_i}\left\{\frac{\nu}{3} D - U - P\right\} dx_i + \oint_C \nu\nabla^2 v_i dx_i + \oint_C d\left(\frac{1}{2} v_i v_i\right)$$

$$= \oint_C d\left\{\frac{\nu}{3} D - U - P\right\} + \nu\oint_C \nabla^2 v_i dx_i + \oint_C d\left(\frac{1}{2} v_i v_i\right)$$

or, $$\frac{d\Gamma}{dt} = \oint_C d\left\{\frac{\nu}{3} D - U - P + \frac{1}{2} v_i v_i\right\} + \nu\oint_C \nabla^2 v_i dx_i$$

$$= \nu \oint_C \nabla^2 v_i dx_i$$ (since $\frac{\nu}{3} D - U - P + \frac{1}{2} v_i v_i$ is a single-valued function, and the integration is "round a closed circuit")

or, $$\frac{d\Gamma}{dt} = \nu\nabla^2 \oint_C v_i dx_i$$

$$= \nu\, \nabla^2 \Gamma \qquad (8.5)$$

It is to be noted that for uniform viscous incompressible fluid we get D = 0, and $\frac{1}{\rho}\frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{p}{\rho}\right)$. Consequently we arrive at the same relation (8.5) for $\frac{d\Gamma}{dt}$

For perfect fluid $\nu = 0$, and therefore $\frac{d\Gamma}{dt} = 0$, that is, Γ = constant in time. This was established earlier, and it is Kelvine's theorem on the constancy of circulation of perfect fluid.

8.9. FLOW BETWEEN PARALLEL PLATES

We consider here a steady two-dimensional laminar flow of an incompressible viscous fluid between two parallel planes. We take the direction of flow as the x_1-axis (Fig 9.1)

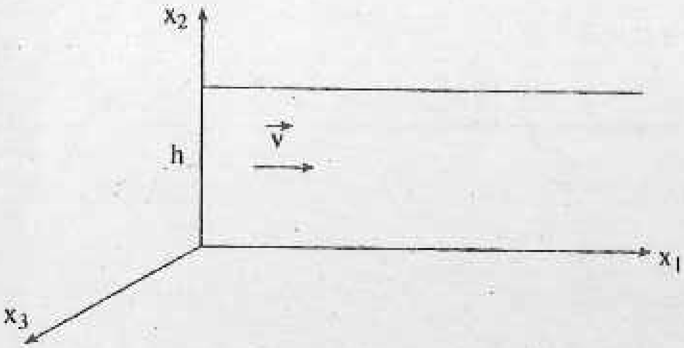


Fig. 9.1

Let $x_2 = 0$ and $x_2 = h$ be the equations of the planes between which the fluid flows under no body forces except the pressure. As the motion is two-dimensional in x_1, x_2 plane, it is the same in each of the layers parallel to x_1, x_2. For this laminar flow we can have

$$v_1 = v_1(x_1, x_2), \quad v_2 = 0 = v_3 \tag{9.1}$$

and

$$p = p(x_1, x_2) \tag{9.2}$$

Since we are considering incompressible fluid, we must have

$$D = \text{div}\,\vec{v} = 0 \quad \text{or,} \quad \frac{\partial v_1}{\partial x_1} = 0, \text{ that is, } v_1 \text{ is independent of } x_1.$$

Therefore, $v_1 = v_1(x_2)$ (9.3)

Now, the Navier Stokes equation of motion for incompressible viscous fluid is

$$\frac{\partial \vec{v}}{\partial t}+\left(\vec{v}.\nabla\right)\vec{v}=\vec{F}-\frac{1}{\rho}\text{grad }p+\nu\nabla^2\vec{v}$$

which for the present case becomes

$$iv_1\frac{\partial v_i}{\partial x_1}=-\frac{1}{\rho}\frac{i\partial p}{\partial x_1}+i\nu\nabla^2 v_1-\frac{1}{\rho}j\frac{\partial p}{\partial x_2} \quad (9.4)$$

where i, j are unit vectors along x_1 and x_2 axes respectively. (since for steady motion $\frac{\partial \vec{v}}{\partial t}=0$, and also there is no body force $\vec{F}$. Also, we have used (9.1), (9.2) and (9.3)).

From (9.4) it follows that

$$v_1\frac{\partial v_1}{\partial x_1}=-\frac{1}{\rho}\frac{\partial p}{\partial x_1}+\nu\nabla^2 v_1 \quad (9.5a)$$

and $$0=-\frac{1}{\rho}\frac{\partial p}{\partial x_2} \quad (9.5b)$$

From (9.5b), we see that $\frac{\partial p}{\partial x_2}=0$, that is, p is independent of x_2. Thus, from (9.2) we have

$$p=p(x_1) \quad (9.6)$$

From (9.5a) we get

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial x_1}+\nu\nabla^2 v_1 \quad \text{(because of (9.3))}$$

or, $$\frac{\partial p}{\partial x_1}=\frac{dp}{dx_1}=\rho\nu\nabla^2 v_1$$

or, $$\frac{dp}{dx_1}=\mu\nabla^2 v_1=\mu\frac{\partial^2 v_1}{\partial x_2^2} \quad \left(\because \nu=\frac{\mu}{\rho}\right) \quad (9.7)$$

Now, the L.H.S. of (9.7) is a function of x_1 alone, whereas the R.H.S. of it is a function of x_2 alone. Therefore, each side must be a constant. Also, since the fluid is moving in the positive x_1 direction, the pressure p should decrease as x_1 increases, that is, $\frac{dp}{dx_1}$ is a negative constant. Therefore,

$$\frac{dp}{dx_1} = \mu \frac{\partial^2 v_1}{\partial x_2^2} = \mu \frac{d^2 v_1}{dx_2} = -C \text{ (say)} \quad (C > 0) \quad (9.8)$$

or, $$\frac{d^2 v_1}{dx_2} = -\frac{C}{\mu}$$

Integrating, we have $v_1 = -\frac{Cx_2^2}{2\mu} + C_1 x_2 + C_2$ (9.9)

where C_1 and C_2 are constants of integration, and may be determined from the boundary conditions. We take the boundary conditions as follows :

Case 1. $v_1 = 0$ at the planes $x_2 = 0$ and $x_2 = h$.

These boundary conditions lead to

$$0 = C_2$$

$$0 = -\frac{Ch^2}{2\mu} + C_1 h + C_2$$

Solving we have $C_1 = \frac{Ch}{2\mu}$, $C_2 = 0$

Therefore, from (9.9) we get v_1 as

$$v_1 = -\frac{Cx_2^2}{2\mu} + \frac{Ch}{2\mu} x_2 = \frac{C}{2\mu} x_2 (h - x_2) \quad (9.10)$$

which is a parabola in $x_1 x_2$-plane. Such a laminar flow is called plane poiseuille flow. The maximum velocity $(v_1)_{max}$ can be obtained from the condition that this maximum velocity occurs where $\frac{dv_1}{dx_2} = 0$, i.e., $h - 2x_2 = 0$ or, $x_2 = \frac{h}{2}$

Consequently, $(v_1)_{max} = \frac{C}{2\mu} \frac{h}{2}\left(h - \frac{h}{2}\right) = \frac{Ch^2}{8\mu}$

Now, the tangential stress at (x_1, x_2) is given by

$$T_{12} = \mu \frac{dv_1}{dx_2} = \frac{C}{2}(h - 2x_2) \quad (9.11)$$

Therefore, the drag per unit area on the lower plane $(x_2 = 0) = \frac{Ch}{2}$

Case-2. Let us now consider a boundary condition that the lower plane is fixed, that is, $v_1 = 0$ at $x_2 = 0$, but the uper upper is in uniform motion with velocity V in x_1 direction, that is, $v_1 = V$ at $x_2 = h$. Therefore, $C_2 = 0$ and $V = -\frac{Ch^2}{2\mu} + C_1 h$

$$\text{or,}\quad C_1 = \frac{V}{h} + \frac{Ch}{2\mu}.$$

$$\text{Consequently,}\quad v_1 = -\frac{Cx_2^2}{2\mu} + x_2\left(\frac{V}{h} + \frac{Ch}{2\mu}\right) \quad \ldots \quad (9.12)$$

Here also, the velocity profile of this flow is a parabola.

This laminar flow between two parallel planes one of which is at rest and other moving uniformly parallel to the fixed plane is known as generalized Couette flow. The maximum velocity occur at $\frac{dv_1}{dx_2} = 0$, or, $-\frac{Cx_2}{\mu} + \left(\frac{V}{h} + \frac{Ch}{2\mu}\right) = 0$

$$\text{or,}\quad x_2 = \frac{h}{2} + \frac{\mu V}{Ch}$$

$$(v_1)_{max} = -\frac{C}{2\mu}\left(\frac{h}{2} + \frac{\mu V}{Ch}\right)^2 + \left(\frac{h}{2} + \frac{\mu V}{Ch}\right)\left(\frac{V}{h} + \frac{Ch}{2u}\right)$$

$$= \frac{\mu V^2}{2Ch^2} + \frac{V}{2} + \frac{Ch^2}{8\mu} \quad (9.13)$$

When the pressure remains constant as x_1 increases, $\frac{dp}{dx_1} = 0$.

Therefore, $C = 0$ (from (9.8)). Then, we have from (9.12)

$$v_1 = \frac{V}{h} x_2 \quad (9.14)$$

In this case, the motion of fluid in x_1-direction is entirely due to the viscous force. Such flow is known as simple Couette flow or plane Couette flow.

Ex.9.1. For generalized Couette flow between two parallel planes find the total flow per unit breadth across a plane perpendicular to x_1-axis, the direction of flow. Also, find the drag per unit area on both the upper and lower planes.

Hints : The total flow per unit breadth across a plane perpendicular to x_1-direction is

$$\int_0^h v_1(x_2)dx_2$$ where $x_2 = 0$ and $x_2 = h$ are the equations of the planes.

Putting $v_1(x_2)$ in this integral it is calculated. Ans. $\frac{1}{2}Vh + \frac{1}{12}\frac{Ch^3}{\mu}$, where V is the velocity of the upper plane.

$$T_{12} = \mu\frac{dv_1}{dx_2}, \quad \text{drag. at lower plane} = \mu\left.\frac{dv_1}{dx_2}\right|_{x_2=0}$$

$$\text{drag. at the upper plane} = \mu\left.\frac{dv_1}{dx_2}\right|_{x_2=h}$$

Ans. $\frac{\mu V}{h} + \frac{1}{2}Ch$ and $\frac{\mu V}{h} - \frac{1}{2}Ch$ respectively.

Unit : 9 ❑ Cartesian Tensors

9.1. TRANSFORMATION OF COORDINATES

We consider two sets of rectangular cartesian axes (Ox_1, Ox_2, Ox_3) and (Ox'_1, Ox'_2, Ox'_3) having the same origin. The coordinates of a point P with respect to these sets of axes are respectively (x_1, x_2, x_3) and (x'_1, x'_2, x'_3). These coordinates are related by the equations

$$\left.\begin{aligned} x'_j &= a_{1j} + a_{2j}x_2 + a_{3j}x_3 \\ &= \sum_{i=1}^{3} a_{ij}x_i \\ (j &= 1, 2, 3) \end{aligned}\right\} \qquad \text{......(1.1)}$$

where a_{ij} is the cosine of the angle between the directions of x_i and x'_j increasing. The inverse relations which expresses the coordinates (x_1, x_2, x_3) in terms of (x'_1, x'_2, x'_3) are

$$\left.\begin{aligned} x_i &= \sum_{j=1}^{3} a_{ij}x'_j \\ (i &= 1, 2, 3) \end{aligned}\right\} \qquad \text{......(1.2)}$$

The relations (1.1) and (1.2) are the equations expressing the transformation of coordinates. These transformation of coordinates can be written with the use of "summation convention". In this convention the summation sign will be automatically understood whenever a suffix is repeated. That is, if a suffix is repeated (i.e., appears twice) it is given all its possible values, and the terms are to be added all. With this convention the transformations of coordinates (1.1) and (1.2) are written as, respectively

$$x'_j = a_{ij}\, x_i \quad (j = 1, 2, 3) \qquad \text{......(1.3)}$$

and $$x_i = a_{ij}\, x'_j \qquad \text{......(1.4)}$$

$$(i = 1, 2, 3)$$

9.2 TENSORS

Let (u_1, u_2, u_3) be a set of three quantities related to the unprimed coordinate system (Ox_1, Ox_2, Ox_3). If (u'_1, u'_2, u'_3) is the corresponding set of quantities related to the primed coordinate system (Ox'_1, Ox'_2, Ox'_3) such that the transformation between these two sets are the same as that between the coordinates with respect to the unprimed and primed coordinate systems, then this set of three quantities is called a tensor of the first order or a vector. Thus, the transformations between u_i $(i = 1, 2, 3)$ and u'_j $(j = 1, 2, 3)$ are given by

$$u'_j = a_{ij} u_i \quad (j = 1, 2, 3) \qquad \text{......(2.1)}$$

$$\text{and} \quad u_i = a_{ij} u'_j \quad (i = 1, 2, 3) \qquad \text{......(2.2)}$$

The individual u_1, u_2, u_3 are called the components of the tensor.

If we multiply u_i and u'_j by the same quantity k then we get a vector or tensor of order one, because

$$k u'_j = a_{ij}(ku_i) \text{ and } ku_i = a_{ij}(k u'_j)$$

By addition or subtraction of two vectors u_i and v_i we get another vector since

$$u'_j \pm v'_j = a_{ij}(u_i \pm v_i)$$

Now, we can multiply two vectors or tensors of order one to get a set of nine quantities $u_i v_j$ $(i = 1, 2, 3 ; j = 1, 2, 3)$. Since each of the vectors u_i and v_j satisfies (2.1) and (2.2) we must have

$$u'_j v'_k = (a_{ij} u_i)(a_{lk} o_l) = a_{ij} a_{lk} (u_i v_l) \qquad \text{......(2.3)}$$

Thus, a set of nine quantites $u_i v_l$ with respect to the unprimed coordinate system is related to the corresponding nine quanties $u'_j v'_k$ with respecttot he primed coordinate system through the transformation property given by (2.3). In general if a set of nine quantites ω_{il} referred to the unprimed system is connected with the corresponding set of nine quantites ω'_{jk} referred to the primed set through the trans formation :

$$\omega'_{jk} = a_{ij} a_{lk} \omega_{il} \qquad \text{......(2.4)}$$

then this set of quantites ω_{il} is called a tanser of second order. Thus, the product of two vectors $u_i\ v_j$ is a tensor of second order. It is to be noted that the coordinates of a point (x_1, x_2, x_3) form a vector, and the product of coordinates x_i and y_j of two points, that is $x_i y_j$ forms a tensor of second order. We can similarly y construct and define tensors of third, fourth, and higher orders. In fact, the tensors of third and fourth orders transform like the products of three vectors $u_j\ v_j\ \omega_k$ and four vectors $u_i\ v_j\ \omega_k\ x_l$ respectively. That is, the tensors of third and fourth orders, ω_{ijk} and ω_{ijkl} transform as follows :

$$\left.\begin{aligned} \omega'_{ijk} &= a_{\alpha i}\, a_{\beta j}\, a_{\gamma k}\, \omega_{\alpha\beta\gamma} \\ \omega'_{ijkl} &= a_{\alpha i}\, a_{\beta j}\, a_{\gamma k}\, a_{\delta l}\, \omega_{\alpha\beta\gamma\delta} \end{aligned}\right\} \quad \ldots\ldots(2.5)$$

In this way, the tensors of higher order can be defined. In the relations (2.1)—(2.5) the suffixes which are repeated are called "dummy suffixes". These suffixes are to be given all possible values and then added all. Therefore, it is unimportant whether one assigns suffix i or j or k. Consequenly, one can interchange any two dummy suffixes in a relation. For example, we can interchange the dummy suffixes i and l in (2.4) to obtain

$$\omega'_{jk} = a_{lj}\, a_{ik}\, \omega_{li} = a_{ik}\, a_{lj}\, \omega_{li} \quad \ldots\ldots(2.6)$$

Therefore, ω_{li} transforms according to the same rule as ω_{il}. Consequently, ω_{li} is another tensor of second order. Again, $\omega_{il} \pm \omega_{li}$ are tensors of second order as these are obtained by addition and subtraction of two tensors of second order. Also, we see that

$$\left.\begin{aligned} \omega_{il} &= \frac{1}{2}\left(\omega_{il} + \omega_{li}\right) + \frac{1}{2}\left(\omega_{il} - \omega_{li}\right) \\ &= u_{il} + v_{il} \end{aligned}\right\} \quad \ldots\ldots(2.7)$$

where $$u_{il} = \frac{1}{2}(\omega_{il} + \omega_{li}) = u_{li} \quad \ldots\ldots(2.8)$$

and $$v_{il} = \frac{1}{2}(\omega_{il} - \omega_{li}) = -v_{li} \quad \ldots\ldots(2.9)$$

The tensor u_{il} is a symmetric tensor of second order, and v_{il} is an antisymmetric or skew-symmetric tensor of second order. Thus, a tensor of second order can be written as the sum of two tensors, one symmetric and the other antisymmetric.

9.3. CONTRACTION AND INNER PRODUCT OF TENSORS

We can make two suffixes in a tensor of any order equal, and consequently add the terms arising out from these repeated suffixes. This opeation is known as contraction. For example, the contraction for the tensor of third order, ω_{ijk}, gives

$$\omega_{iik} = \omega_{11k} + \omega_{22k} + \omega_{33k} \qquad \text{......(3.1)}$$

which is a tensor of first order, that is, a vector. To prove this, we recall the transformation property of w_{ijk}, which is

$$\omega'_{ijk} = a_{i'i}\, a_{j'j}\, a_{k'k}\, \omega_{i'j'k'}$$

Therefore , $\omega'_{iik} = a_{i'i}\, a_{j'i}\, a_{k'k}\, \omega_{i'j'k'} = a_{kk'}(a_{i'i}\, a_{j'i}\, \omega_{i'j'k'})$

Now, $a_{i'i}$ are the direction consines of the axis of $x_{i'}$ with respect to the x'_i, and $a_{j'i}$ are these of $x_{j'}$ with respect x'_i. Therefore, $a_{i'i}\, a_{j'i}$ is the consine of the angle between $x_{i'}$ and $x_{j'}$, and is equal to 1 if the axes are identical and to 0 if they are perpendicular. Thus

$$a_{i'i}\, a_{j'i} = \delta_{i'j'} \qquad \text{......(3.2)}$$

where $\delta_{i'j'}$ is the Kronecker delta, given by

$$\left.\begin{aligned} \delta_{i'j'} &= 1 \text{ if } i' = j' \\ &= 0 \text{ if } i' \neq j' \end{aligned}\right\} \qquad \text{......(3.3)}$$

with this, we have from above

$$\omega'_{iik} = a_{k'k}\left(\delta_{i'j'}\, \omega_{i'j'k'}\right) = a_{k'k}\, \omega_{i'i'k'},$$

that is, $\omega_{i'i'k'}$ has the transformation property of a vector.

In general, on contraction from a tensor we get a new tensor whose order is less by two than that of the original tensor.

Exercise 9.3.1. Prove the above assertion.

By contraction of the tensor $u_i\, v_j$, we get the scalar product of the vectors u_i and v_j, that is,

$$u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3 \qquad \text{......(3.4)}$$

The products of tensors like $u_i\ v_j$, $u_i\ v_j\ \omega_k\ x_l$, $\omega_{ij}\ v_k$, $\omega_{ijk}\ \omega_{mn}$ etc. which gives rise to new tensors, are called the "outer product" of the tensors. If we make contraction on these tensors obtained by outer products, we get the inner product of the tensors. For example, the outer product of two tensors ω_{ijk} and ω_{mn} is $\omega_{ijk}\ \omega_{mn}$, and on contraction we get the inner product of these tensor as

$$\omega_{ijk}\ \omega_{kn} = v_{ijn} \quad \text{(say)} \qquad \ldots\ldots(3.5)$$

In general, inner product of two tensors of any order is given as

$$\omega_{ijk\ldots\ldots}\quad \omega_{ij'k'\ldots\ldots} = v_{jk\ldots\ldots j'k'\ldots\ldots} \qquad \ldots\ldots(3.6)$$

9.4. QUOTIENT LAW

Let ω_{ij} be a set of nine quantities. We make inner product of ω_{ij} with any vector u_i to obtain $\omega_{ij}\ u_i = v_j$ (say). If v_j is a vector then we can prove that ω_{ij} is a tensor of second order. In general, if the inner product of a set of 3^n quantities $\omega_{i_1 i_2 \ldots i_n}$ with any vector u_i gives a tensor of $(n-1)$ h order, then $\omega_{i_1 i_2 \ldots i_n}$ is a tensor of n-th order. This is known as quotient law.

Proof : Since $w_{i_1 i_2 \ldots\ldots i_n}\ u_{i_1} \equiv v_{i_2 \ldots i_n}$ is a tensor of $(n-1)$th order, we have

$$v'_{i_2 \ldots\ldots i_n} = a_{j_2 i_2}\ a_{j_3 i_3} \ldots\ldots a_{j_n i_n}\ v_{j_2 j_3 \ldots\ldots j_n}$$

$$\text{or,}\quad w'_{i_1 i_2 \ldots\ldots i_n}\ u'_{i_1} = a_{j_2 i_2} a_{j_3 i_3} \ldots a_{j_n i_n}\ w_{j_1 j_2 \ldots j_n}\ u_{j_1} \quad (4.1)$$

Since u_{j_1} is a vector we must have

$$u'_{i_1} = a_{l i_1}\ u_l$$

Multiplying by $a_{j_1 i_1}$ and summing over i_1 we have

$$a_{j_1 i_1} u'_{i_1} = a_{j_1 i_1} a_{l i_1} u_l = \delta_{j_1 l}\ u_l \quad \text{(using (3.2))}$$

$$= u_{j_1} \qquad (4.2)$$

Therefore, from (4.1) and (4.2) we have

$$w'_{i_1 i_2 \ldots i_n}\ u'_{i_1} = a_{j_2 i_2}\ a_{j_3 i_3} \ldots\ldots a_{j_n i_n} \cdot w_{j_1 j_2 \ldots\ldots j_n} \cdot a_{j_1 i_1}\ u'_{i_1}$$

$$\text{or,}\quad \left\{ w'_{i_1 i_2 \ldots\ldots i_n} - a_{j_1 i_1}\ a_{j_2 i_2} \ldots a_{j_n i_n}\ w_{j_1 j_2 \ldots j_n} \right\} u'_{i_1} = 0$$

Since u_j is any vector, we must have

$$w'_{i_1 i_2 \ldots i_n} = a_{j_1 i_1} a_{j_2 i_2} \ldots a_{j_n i_n} \cdot w_{j_1 j_2 \ldots j_n} \quad (4.3)$$

Therefore, $w_{j_1 j_2 \ldots\ldots j_n}$ is a tensor of n-th order.

9.5. SPECIAL TYPES OF TENSORS

In the above sections we have introduced symmetric and skew-symmetric tensors. Also, we have introduced Kronecker delta, and an important relation (3.2). Actually, by using the relation (3.2) one can arrive at the inverse relation (1.4) from the co-ordinate transformation (1.3). If we multiply both the sides of (1.3) by a_{kj} and sum over j, we get

$$a_{kj} x'_j = a_{kj}\, a_{ij}\, x_i = \delta_{ki}\, x_i \quad \text{(using (3.2))}$$

Therefore, $x_k = a_{kj} x'_j$

which is the inverse transformation formula. It is easy to see that Kronecker delta δ_{ij} is a tensor of second order. In fact,

$$a_{ii'}\, a_{jj'}\, \delta_{ij} = a_{ii'} \left(a_{jj'}\, \delta_{ij} \right) = a_{ii'}\, a_{ij'} \qquad (5.1)$$

Now, $a_{ii'}$ are the direction cosines of the axis $x'_{i'}$, with respect to the axis x_i, and $a_{ij'}$ are those of $x'_{j'}$, with respect x_i. Therefore, $a_{ii'}\, a_{ij'}$ is the cosine of the angle between $x'_{i'}$ and $x'_{j'}$. It is, thus, equal to 1 if $i' = j'$ and to 0 if $i' \neq j'$, that is if the axes are perpendicular, Therefore

$$a_{ii'}\, a_{ij'} = \delta_{i'j'} \qquad (5.2)$$

Using (5.2), we have, from (5.1),

$$\delta_{i'j'} = a_{ii'}\, a_{jj'}\, \delta_{ij} \qquad (5.3)$$

Thus, δ_{ij} has the transformation property of a tensor of second order. The Kronecker delta is, therefore, a tensor of second order. Also, it is to be noted that the component, of this tensor retain the same value in the transformation of coordinate axes, that is, whenever the axes are rotated. Such tensors are called isotropic tensors.

We now define a set of quantities $\in_{ikm}$ with the condition that if any two of i, k, m are equal the corresponding component is 0 ; the components will be + 1 or – 1 if i, k, m are unequal and their order is cyclic or not cyclic respectively. We can examine whether $\in_{ikm}$ is a tensor of third order. If it is such tensor then we must have

$$\in'_{jln} = a_{ij}\, a_{kl}\, a_{mn} \in_{ikm}$$

$$= a_{1j}\, a_{2l}\, a_{3n} + a_{2j} a_{3l} a_{1n} + a_{3j} a_{1l} a_{2n}$$

$$- a_{2j}\, a_{1l}\, a_{3n} - a_{3j}\, a_{2l}\, a_{1n} - a_{1j} a_{3l} a_{2n} \qquad (5.4)$$

Obviously, if any two of j, l, n, are equal, then R.H.S. of (5.4) is zero, and thus $\in'_{jln} = 0$. If j, l, n are all unequal, we have

$$\left.\begin{aligned} \in'_{jln} &= \begin{vmatrix} a_{1j} & a_{2j} & a_{3j} \\ a_{1l} & a_{2l} & a_{3l} \\ a_{1n} & a_{2n} & a_{3n} \end{vmatrix} \\ &= 1 \text{ if } j, l, n \text{ are in cyclic order} \\ &= -1 \text{ if } j, l, n \text{ are not in cyclic order} \\ &= \in_{jln} \end{aligned}\right\} \qquad (5.5)$$

Thus, the set of quantities $\in_{jln}$ is transformed into itself by the rule for transformation of a tensor of third order, and therefore it is a tensor of order 3., This tnesor is known as alternating tensor. This is also isotropic.

Ex.9.5.1. Prove that the determinant $\begin{vmatrix} a_{1j} & a_{2j} & a_{3j} \\ a_{1l} & a_{2l} & a_{3l} \\ a_{1n} & a_{2n} & a_{3n} \end{vmatrix}$ is equal to 1 if j, l, n are in cyclic order, and to – 1 if j, l n are not in cyclic order.

Hints. Use the relations (3.2) and (5.2) to prove that $\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = 1.$

Then, the result follows because the sign of the determinant is reversed if we interchange any two rows of it. As such interchange makes the change in the order of j, l, n from cyclic to non-cyclic, i.e., from even permutation to odd permutation of the suffixes, then the value of the determinant which is + 1 for even permutation changes to – 1 for odd permutation

Now the inner product of the tensors ϵ_{ikl} and u_n is $\epsilon_{ikm}\, u_m = w_{ik}$ (say) which is a tensor of second order. The components of w_{ik} are

$w_{12} = \epsilon_{12m}\, u_m = \epsilon_{123}\, u_3 = u_3, w_{21} = \epsilon_{21m}\, u_m = \epsilon_{213}\, u_3 = -u_3$

$w_{11} = \epsilon_{11m}\, u_m = 0$, and so on.

Therefore, writing these components as the elements of the matrix $\{w_{ij}\}$ we have

$$\{w_{ij}\} = \begin{pmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{pmatrix} \qquad (5.6)$$

The antisymmetric tensor w_{ij} is, thus, associated with the vector u_i, and this antisymmetric tensor appears as the inner product of the alternating tensor with that vector.

Ex. 9.5.2. Prove that

$$\epsilon_{iks}\, \epsilon_{mps} = \delta_{im}\delta_{kp} - \delta_{ip}\delta_{km} \qquad (5.7)$$

Hints. $\epsilon_{iks}\, \epsilon_{mps}$ is the inner product of two tensors of third order, and therefore it is a tensor of fourth order.

If $i = k$, the L.H.S. of (5,7) is zero

If $m = p$, it is also zero

If $i = m$ and $k = p$, then $\epsilon_{iks} = \epsilon_{mps} = \pm 1$ $(i \neq k)$; therefore the component is + 1

But if $i = m$ and $k \neq p$, it is zero

Similarly, if $i = p$ and $k = m$, the component = – 1, it is zero if $k \neq m$.

The above results are also true for the R.H.S. of (5.7). Hence the proof.

Ex.9.5.3. Prove that $\epsilon_{iks}\epsilon_{mks} = 2\delta_{im}$.

Hints : Use (5.7)

Now, let us see the transformation property of the operator $\frac{\partial}{\partial_{\cdot i}}$ $(i = 1, 2, 3)$ under co-ordinate transformation.

we have $$\frac{\partial}{\partial x_i'} = \frac{\partial}{\partial x_j}\frac{\partial x_j}{\partial x_i'} \qquad (5.8)$$

From (1.4), we have $x_j = a_{ji} x_i'$

Therefore, $$\frac{\partial x_j}{\partial x_i'} = a_{ji} \qquad (5.9)$$

Putting this into (5.8) we get

$$\frac{\partial}{\partial x_i'} = a_{ji} \frac{\partial}{\partial x_j} \qquad (5.10)$$

Thus, the operators $\frac{\partial}{\partial x_i}$ form of tensor of first order or vector. Let ϕ be a scalar (scalar is a quantity which remains invariant under co-ordinate transformation). Then, the gradient is $\frac{\partial \phi}{\partial x_i}$, and we see from (5.10) that $\frac{\partial \phi}{\partial x_i}$ is a tensor of first order or a vector.

It is to be noted that on contraction we get a scalar from a tensor of second order, that is,

$$w_{ii} = w_{11} + w_{22} + w_{23} \text{ is a scalar}$$

Now, if u_i is a vector, than $\frac{\partial u_i}{\partial x_j}$ is a tensor of second order, because it is the product of two vectors $\frac{\partial}{\partial x_j}$ and u_i. On contraction, we get

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \qquad (5.11)$$

which is the divergence of u_i. Note that by forming the product of two vectors u_i and v_j and then on contraction we get the scalar product of u_i and v_j, that is,

$$u_i\, v_i = u_1 v_1 + u_2 v_2 + u_3 v_3$$

We have seen earlier that a tensor of 2nd order can be written as the sum of two tensors of 2nd order, one of which is symmetric and the other is skew-symmetric. Therefore, the second order tensor $\frac{\partial u_i}{\partial x_j}$ can be written as

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \qquad (5.12)$$

The tensor of 2nd order, $\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}$ (which is skew-symmetric), is known as "curl" or "rotation" of u_i.

মানুষের জ্ঞান ও ভাবকে বইয়ের মধ্যে সঞ্চিত করিবার যে একটা প্রচুর সুবিধা আছে, সে কথা কেহই অস্বীকার করিতে পারে না। কিন্তু সেই সুবিধার দ্বারা মনের স্বাভাবিক শক্তিকে একেবারে আচ্ছন্ন করিয়া ফেলিলে বুদ্ধিকে বাবু করিয়া তোলা হয়।

— রবীন্দ্রনাথ ঠাকুর

"Any system of education which ignores Indian conditions, requirements, history and sociology is too unscientific to commend itself to any rational support".

— Subhas Chandra Bose

ভারতের একটা mission আছে, একটা গৌরবময় ভবিষ্যৎ আছে, সেই ভবিষ্যৎ ভারতের উত্তরাধিকারী আমরাই। নূতন ভারতের মুক্তির ইতিহাস আমরাই রচনা করছি এবং করব। এই বিশ্বাস আছে বলেই আমরা সব দুঃখ কষ্ট সহ্য করতে পারি, অন্ধকারময় বর্তমানকে অগ্রাহ্য করতে পারি, বাস্তবের নিষ্ঠুর সত্যগুলি আদর্শের কঠিন আঘাতে ধূলিসাৎ করতে পারি।

— সুভাষচন্দ্র বসু

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